11th World Congress on Structural and Multidisciplinary Optimisation
07th -12th, June 2015, Sydney Australia

Design and optimization of a variable stiffness composite laminate

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1. Abstract
With the wide use of fiber-reinforced composite laminates in aerospace industry, structural instability is one of the most key aspects for flight vehicles. Hence, the research on structural instability mechanisms of composite laminates, especially buckling characteristics, is of great significance. In this paper, the buckling characteristics analysis and design optimization of the variable stiffness composite (VSC) panel with given geometry and material properties are studied. Pagano's three layer constant stiffness laminate is investigated as the baseline case. The method of layered modeling and numerical structural analysis of VSC panel in the commercial package ABAQUS is described. The design optimization of the VSC panel aims to maximize the critical buckling load with the fiber orientation angles at the center and end as design variables, which is solved by the sequential quadratic programming method. It is found that with the same weight, the optimal critical buckling load of the designed VSC panel is increased up to 67.9% compared to the baseline laminate. To further verify the accuracy of the optimal design solution, based on the first von Karman equation, critical buckling load differential equation of a rectangular laminate is deduced, and the fiber orientation angle corresponding to the optimal critical buckling load is derived by theoretical analysis, which is compared with the optimal critical fiber angle by optimization. It is noticed that the two critical fiber angles show great agreement to each other. Therefore, the finite element analysis in conjunct with optimization method is proved to be accurate for solving buckling issues of VSC laminates, which thus could be used in further study of the performance of VSC laminates. It is concluded that, compared to the traditional fiber-reinforced composite laminates, the VSC laminates can be designed with more flexibility to achieve better load redistribution, resulting in great improvements of the structural performance.
2. Key words: variable stiffness composites; buckling analysis; design optimization; finite element analysis.

3. Introduction
Research on application of new material with superior performance is always a hot topic in aerocraft structural design due to the high demand for lightweight structures. Fiber reinforced composite material has many advantages, such as high specific strength and modulus, designable performance and excellent integral forming capability, etc, which almost determines the structure performances of aerocraft. Application of fiber reinforced composite material to aerocraft structure can significantly reduce the weight, meanwhile improve the aerodynamic and flight performances [1].
In order to improve the structural performances and lower the cost during the life cycle, it is necessary to carry out design of composite material, of which one focus is to fully explore the directionality of composites and the designable capability of the structural performance. It is possible to take full advantage of the directionality by using variable stiffness composites (VSC)[2], in which the fiber can be steered in plane with the fiber angle as a function of space. The continuously changed fiber orientation results in the variation of stiffness at different locations, based on which the designers can adjust the distribution of internal load to improve the structural performance or reduce weight [3].
Gurdal and Olmedo [4,5] proposed a fiber placement method by changing the fiber orientation angle along the coordinate axis. They analyzed the in-plane and buckling response of variable stiffness composite panels by closed analytical and numerical methods. The VSC showed considerable improvements in load-carrying capabilities compared to traditional composite laminates. Gurdal and Tatting [6] studied the effects of stiffness variation on the in-plane and buckling response for variable stiffness composite panels. The curvilinear fiber paths for fiber-steered variable stiffness laminates were designed and optimized based on lamination parameters distribution by Setoodeh [7]. Tatting and Gurdal carried out the design and manufacture of tailored tow placed plates, in which the buckling characteristics were studied by both finite element analysis and experimental method [8,9]. Analysis and optimization of variable angle tow composite plates for pre-buckling [10], buckling [11] and post-buckling behavior [12] have been done with differential quadrature method by Raju and Wu.
The structural instability is one of the important issues that impact the structural performance in the application of composite materials, causing structural deformation, decreased load-carrying capacity and even structural damage. Therefore, it is necessary to study the structural instability of composite laminates, such as buckling characteristics. In this paper, the buckling characteristics are designed and optimized for variable stiffness composite laminates, in which the fiber orientation angle linearly varies along the direction of the load. Structural analysis is done in the finite element software ABAQUS. Sequential quadratic programming algorithm (SQP) is

used to optimize the design. In the following, firstly the process of FEA modeling is described. Then theoretical analysis method for the fiber orientation angle corresponding to the critical buckling load is given. Furthermore the design problem for variable stiffness composite laminate is defined. Finally the results of optimization are presented.

4. FEA modeling of variable stiffness composite laminate

Fibers in every single ply are straight for conventional composite laminates, which therefore can be designed only by changing the stacking sequence. However, fibers can be steered in plane to make the fiber orientation angle continuously vary with spacial location for variable stiffness composite laminates, which makes its stiffness as a function of spatial location.

The scheme of the reference path of steered fiber proposed by Gurdal [2] is used in this paper. The fiber orientation angle linearly varies along the direction of loading, and function of fiber orientation angle of the reference path is as follows:

$$\theta(x) = \frac{2(T_1 - T_0)}{a}|x| + T_0 \tag{1}$$

Eq.(1) is written as $\theta(x) = k|x| + b$ for convenience, where $k = 2(T_1 - T_0)/a$ and $b = T_0$. Then the reference path of curve fiber can be defined as:

$$y = \begin{cases} \frac{1}{k}\left\{\ln\left[\cos(kx+b)\right] - \ln\left[\cos(b)\right]\right\} & -\frac{a}{2} \le x < 0 \\ -\frac{1}{k}\left\{\ln\left[\cos(kx+b)\right] - \ln\left[\cos(b)\right]\right\} & 0 < x \le \frac{a}{2} \end{cases} \tag{2}$$

Pagano's three layer case [13] is studied as the reference case The panel is square with cross-ply [0/90/0] stacking sequence. Its side length is $a = 50\text{mm}$ and thickness is $h = 1\text{mm}$. The variable stiffness cases studied in this paper are also stacked in a cross-ply $\left[\langle(T_0 - 90)\,|\,(T_1 - 90)\rangle/\langle T_0\,|\,T_1\rangle/\langle(T_0 - 90)\,|\,(T_1 - 90)\rangle\right]$ sequence, where the function of fiber orientation angle is $\theta(x) = k|x| + b$ in the middle ply, and $\theta(x) = k|x| + b - 90$ in the top and bottom ply.

The ABAQUS user subroutine ORIENT is employed to implement the different orientation to fibers within the plane of each ply in variable stiffness composite laminate. 8-nodec elements C3D8I with incompatible modes are used to establish FEA model for each single ply of laminates, respectively. The panel is simply supported at all the four sides, and it carries unidirectional compression load. The same carbon fiber/epoxy resin material with the reference case is used, and its performance parameters are listed in Table 1. The subscripts 1, 2 and 3 represent fiber direction, transverse direction and thickness direction respectively.

Table 1: Performance parameters of carbon fiber/epoxy resin material

E_{11}[GPa]	E_{22}[GPa]	E_{33}[GPa]	G_{12}[GPa]	G_{13}[GPa]	G_{23}[GPa]	ν_{12}	ν_{13}	ν_{23}
25	1.0	1.0	0.5	0.5	0.2	0.25	0.25	0.25

5. Theoretical analysis of the critical buckling load

The first von Karman equation for large deflections is as follows [16]:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2} - p_x\frac{\partial w}{\partial x} - p_y\frac{\partial w}{\partial y} + p_z \tag{3}$$

For laminate under unidirectional in-plane loading, the governing equation is obtained by setting $N_{xy} = N_y = p_x = p_y = p_z = 0$:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = N_x\frac{\partial^2 w}{\partial x^2} \tag{4}$$

where w is the out-of-plane displacement of the laminate.

The laminate is assumed to be simply supported all around its boundary and the only load is N_x. Then the boundary conditions are:

$w = M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} = 0$ at x=0 and x= a; $w = M_y = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} = 0$ at y=0 and y= a.

where a is the length of the square laminate. In order to satisfy all of the boundary conditions, the expression for w is assumed as:

$$w = \sum\sum A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} \tag{5}$$

Substituting Eq.(5) into Eq.(4), one has

$$\pi^2 A_{mn}\left[D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2 + D_{22}n^4\right] = -A_{mn}a^2N_xm^2 \tag{6}$$

The out-of-plane displacement w is nonzero when buckling occurs, indicating that the cofficient A_{mn} is nonzero thus can be neglected in Eq.(6),

$$\pi^2\left[D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2 + D_{22}n^4\right] = -a^2N_xm^2 \tag{7}$$

For convenience, the buckling load is denoted by $N_o = -N_x$, where the minus sign indicates compression. Then we get the buckling load as:

$$N_o = \frac{\pi^2\left[D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2 + D_{22}n^4\right]}{a^2m^2} \tag{8}$$

where m is the number of half-waves in the x direction and n is the number of half-waves in the y direction, which define the buckling mode. The buckling load N_o is a function of m and n, and changes with m and n. The critical buckling load is the lowest value of Eq.(8), so it is necessary to minimize the right hand side of Eq.(8) with respect to m and n. Obviously, this minimum is achieved when n=1, which means that there is only one half-wave in the transverse direction of the applied loading. Substituting n=1 into Eq.(8), one obtains,

$$N_o = \frac{\pi^2}{a^2}\left[D_{11}m^2 + 2(D_{12} + 2D_{66}) + \frac{D_{22}}{m^2}\right] \tag{9}$$

The buckling load of a simply supported square composite laminate under unidirectional compression load can be obtained through minimizing Eq.(9) with respect to m. The partial derivative with respect to m is

$$\frac{\partial N_o}{\partial m} = \frac{2\pi^2}{a^2}\left[D_{11}m - \frac{D_{22}}{m^3}\right] = 0 \tag{10}$$

Since $\frac{\pi^2}{a^2} \neq 0$, the buckling load is obtained when $D_{11}m - \frac{D_{22}}{m^3} = 0$. The bending stiffness matrix [D] is

$$D = \frac{h^3}{12}\left(\Gamma_0 + \Gamma_1V_{1D} + \Gamma_2V_{2D} + \Gamma_3V_{3D} + \Gamma_4V_{4D}\right) \tag{11}$$

and the corresponding laminate parameters are

$$\left(V_{1D}, V_{2D}, V_{3D}, V_{4D}\right) = 12\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{z}^2\left(\cos 2\theta(\xi), \sin 2\theta(\xi), \cos 4\theta(\xi), \sin 4\theta(\xi)\right)d\bar{z} \tag{12}$$

Then the laminate parameters of the three layer cross-ply laminates in this paper are expressed as:

$$V_{1D} = 12\left[\int_{-\frac{1}{2}}^{-\frac{1}{6}} \bar{z}^2\cos 2\theta(\xi)d\bar{z} + \int_{-\frac{1}{6}}^{\frac{1}{6}} \bar{z}^2\cos 2\left(\theta(\xi) + 90\right)d\bar{z} + \int_{\frac{1}{6}}^{\frac{1}{2}} \bar{z}^2\cos 2\theta(\xi)d\bar{z}\right] = \frac{25}{27}\cos 2\theta(\xi) \tag{13}$$

$$V_{2D} = 12\left[\int_{-\frac{1}{2}}^{-\frac{1}{6}} \bar{z}^2\sin 2\theta(\xi)d\bar{z} + \int_{-\frac{1}{6}}^{\frac{1}{6}} \bar{z}^2\sin 2\left(\theta(\xi) + 90\right)d\bar{z} + \int_{\frac{1}{6}}^{\frac{1}{2}} \bar{z}^2\sin 2\theta(\xi)d\bar{z}\right] = \frac{25}{27}\sin 2\theta(\xi) \tag{14}$$

$$V_{3D} = 12\left[\int_{-\frac{1}{2}}^{-\frac{1}{6}} \bar{z}^2\cos 4\theta(\xi)d\bar{z} + \int_{-\frac{1}{6}}^{\frac{1}{6}} \bar{z}^2\cos 4\left(\theta(\xi) + 90\right)d\bar{z} + \int_{\frac{1}{6}}^{\frac{1}{2}} \bar{z}^2\cos 4\theta(\xi)d\bar{z}\right] = \cos 4\theta(\xi) \tag{15}$$

$$V_{4D} = 12\left[\int_{-\frac{1}{2}}^{-\frac{1}{6}} \bar{z}^2\sin 4\theta(\xi)d\bar{z} + \int_{-\frac{1}{6}}^{\frac{1}{6}} \bar{z}^2\sin 4\left(\theta(\xi) + 90\right)d\bar{z} + \int_{\frac{1}{6}}^{\frac{1}{2}} \bar{z}^2\sin 4\theta(\xi)d\bar{z}\right] = \sin 4\theta(\xi) \tag{16}$$

Substituting these laminate parameters into Eq.(11), the cofficients of bending stiffness matrix are obtained as follows:

$$D_{11} = \frac{h^3}{12}\left[U_1 + \frac{25}{27}U_2\cos 2\theta(\xi) + U_3\cos 4\theta(\xi)\right] \tag{17}$$

$$D_{22} = \frac{h^3}{12}\left[U_1 - \frac{25}{27}U_2\cos 2\theta(\xi) + U_3\cos 4\theta(\xi)\right] \tag{18}$$

Substituting D_{11} and D_{22} into Eq.(10), one has,

$$m\left[U_1 + \frac{25}{27}U_2\cos 2\theta(\xi) + U_3\cos 4\theta(\xi)\right] - \frac{1}{m^3}\left[U_1 - \frac{25}{27}U_2\cos 2\theta(\xi) + U_3\cos 4\theta(\xi)\right] = 0 \tag{19}$$

Substituting m (the number of half-waves in the x direction) into Eq.(19), the fiber orientation angle

$q(x)$ corresponding to the critical buckling state can be obtained.

6. Design and optimization of variable stiffness composite laminate

6.1 Optimization model and method

The slope and initial value of the function of fiber orientation angle are considerd as the design object in the optimization model for the variable stiffness composite laminates, and the objective is to achieve the maximum buckling load. Because both the slope and initial value are determined by the fiber angle at the center point of the panel denoted as T_0 and the fiber angle at the end of the panel denoted as T_1, T_0 and T_1 are treated as design variables. The mathematical model of the optimization of variable stiffness composite laminate is defined as:

$$\begin{aligned} \max \quad & f(X) \\ \text{s.t.} \quad & -90 \le X_1 \le 90 \\ & -90 \le X_2 \le 90 \end{aligned} \tag{20}$$

where design variables X_1 and are T_0 and T_1, respectively. The objective function $f(X)$ presents the eigenvalues of linear buckling analysis for the variable stiffness composite laminate. The constraint conditions are the ranges of the value of T_0 and T_1.

Since the design variables are continuous, the gradient-based sequential quadratic programming (SQP) is used in this optimization. Because SQP is inclined to converge to local minima near the initial value, optimization from multiple initial points are made in this paper in order to increase the possibility to get the optimal design.

User subroutines are used to define the fiber orientation angle of the variable stiffness composite laminate. Thus it is necessary to simultaneously parameterize the FEA model and the user subroutines. For each iteration during the optimization process, the design variables are firstly transferred to the user subroutine, in which relevant parameters are modified, a new user subroutine is generated and then implements different fiber orientation angles to the FEA model to complete the buckling analysis.

6.2 Optimization results

A variable stiffness composite laminate is optimized using SQP in this paper. Table 2 shows the critical buckling load of constant stiffness composite (CSC) laminate, the initial and optimal variable stiffness composite (VSC) laminate respectively. Since fiber orientation angle of each single layer in CSC laminate is constant, the CSC laminate with the stacking sequence [0|0|90] is obtained by setting $T_0 = T_1 = 0$. It can be seen from Table 2 that there are two optimal design points as X_1^*<35.1|50.6> and X_2^*<-35.1|-50.6> in this optimization, with the same critical buckling load as 58.291MPa, which is improved up to 67.9% and 182.5% compared with that of the CSC reference case (34.717MPa) and the initial VSC baseline (20.632MPa). And the reason is that the functions of fiber orientation angles corresponding to these design points are as:

$$X_1^*: \quad q_1(x) = k_1 x + b_1 = \frac{50.6 - 35.1}{25} x + 35.1 = 0.62x + 35.1 \tag{21}$$

$$X_2^*: \quad q_2(x) = k_2 x + b_2 = \frac{-50.6 + 35.1}{25} x - 35.1 = -0.62x - 35.1 \tag{22}$$

It is obvious that $q_2(x) = -q_1(x)$, which means fiber orientation angles at all points in the fiber paths defined by these two functions will have the same magnitude but opposite direction, if their coordinate value at x-direction are the same. The VSC panel is square, so its structure is symmetric with respect to the x-axis. The structural performance are the same when fiber orientation angles have the same magnitude and opposite direction. Thus there definitely exist two optimal solutions with equal magnitude but opposite directions.

Table 1: Optimization results of VSC laminate

	CSC	VSC initial	Optimum X_1^*	Optimum X_2^*
X_1	0	0	35.1	-35.1
X_2	0	15	50.6	-50.6
f(X)	34.717	20.632	58.291	58.291

The variation of critical buckling load of VSC laminates with the slope and initial value of the function of fiber orientation angle are further studied in the optimization process, shown in Figure 1. There are two maximum peaks and two second largest peaks with the double-saddle distribution.

It can be seen from the projected contour that both these two pairs of peaks are symmetric with respect to the Point (0, 0), which demonstrates that the critical buckling load is a binary function of the slope of the initial value, and also confirms the results of the above analysis that there are two optimal solutions with equal magnitude and opposite directions. When the initial value is kept constant, the critical buckling load varies with the slop similar to an M shape, and the maximum value in each group increases substantially with the increasement of the absolute value of the initial value and achieves maximum when the initial value is in the vicinity of ± 45 °. Meanwhile, when the slop is kept constant, the critical buckling load also varies with the initial value similar to an M shape, and the maximum value in each group increases substantially with the decreasement of the absolute value of the slop and achieves maximum when the slop is in the vicinity of ± 0.5.

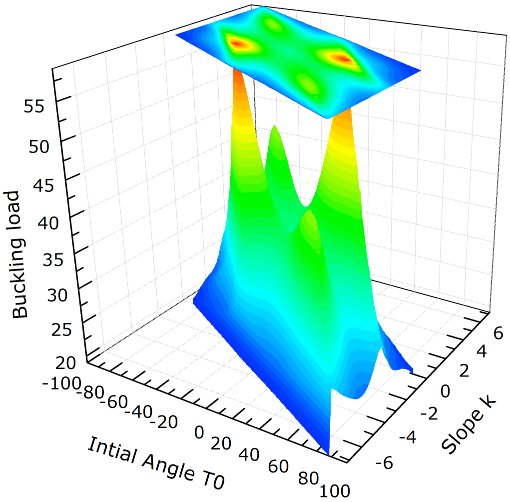
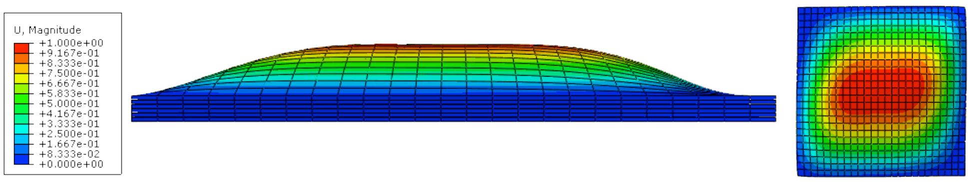
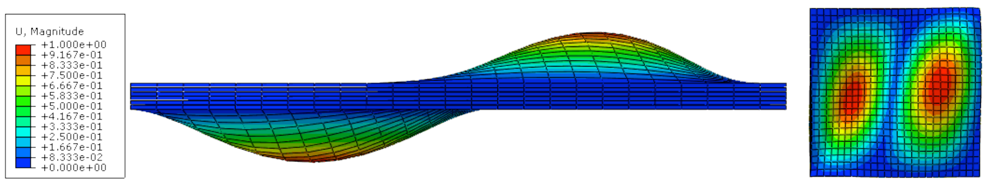


Figure 1: Variation of critical buckling load of VSC laminates with the slope and initial value of the function of fiber orientation angle



(a) First buckling mode



(b) Second buckling mode

Figure 2: The buckling modes and displacement distribution of the optimal design

Figure 2 shows the X-Y and X-Z views of the first and second buckling modes and displacement distribution of the optimal design X_1^*<35.1|50.6>, which are significantly different from each order.

It is noticed from Figure 2(a) that there is one half-wave in the x-direction in the first mode, which means m=1, and the maximum displacement occurs at the center area of the panel. Substituting m=1 into Eq.(19), one has,

$$\left[U_1 + \frac{25}{27}U_2\cos 2\theta(x) + U_3\cos 4\theta(x)\right] - \left[U_1 - \frac{25}{27}U_2\cos 2\theta(x) + U_3\cos 4\theta(x)\right] = \frac{50}{27}U_2\cos 2\theta(x) = 0 \quad (23)$$

Since U_2 is invariant determined by material characteristics and is a non-zero constant, it can be neglected from Eq.(23), then

$$\cos 2\theta(x) = 0 \quad (24)$$

Then the theoretical fiber angle corresponding to the critical buckling load is obtained as $\theta(x) = \pm 45°$. The functions of the fiber orientation angle corresponding to the two optimal design points are respectively $\theta_1(x) = 0.62x + 35.1$ and $\theta_2(x) = -0.62x - 35.1$. The maximum stresses of the first mode occurs in the central region of the panel, and the corresponding range for fiber angle is $\theta \in [39°, 46.8°]$ approximately. Clearly, the result of the theoretical model locates within this range. It indicates that the finite element analysis method and design methodology used in this paper are reasonable and accurate.

7. Conclusions

A variable stiffness composite laminate was optimized for maximum buckling load carrying capability. Its buckling characteristics were analyzed by both theoretical derivation and finite element analysis method. The optimal design under unidirectional compression loading was obtained, with the improvement up to 67.9% compared with the reference constant stiffness case.

The buckling characteristics of variable stiffness composite laminates have close relationship with the initial value and the slope of the function of fiber orientation angle, with which the value of critical buckling load, the shape of buckling modes and the distribution of displacement have great variation.

The fiber orientation angle of the optimal design shows great agreement with that of the theoretical model. It verifies the reasonability and accuracy of the combined solution of the finite element analysis method and design methodology, which can be used in future research of variable stiffness composite materials.

Compared with constant stiffness composite laminates, variable stiffness composite materials with steered fibers have the advantage of high design flexibility. The reference fiber path can be designed according to specific mission requirements, in order to achieve redistribution of loads and thus improve the performance of whole structure.

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