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A TIMP Method for Topology Optimization with Displacement and Stress Constraints in Multiple Loading Cases

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1. Abstract
A method of transplanting ICM (Independent Continuous and Mapping) ideas into material with penalization (TIMP) for continuum structural topology optimization is proposed in this paper. TIMP method is a development of SIMP (Solid Isotropic Material with Penalization) method, which is widely studied and used by internal and overseas researchers. Since the filter function in ICM and the penalty function in SIMP are observed regarding to their similar formulations, the mathematical connection between the two methods yields analogies. Thus, several progresses in ICM are transplanted into SIMP for further developments, which yield to the TIMP method. There are two basic perspectives in TIMP: (1) weight and allowable stress penalty functions are added into SIMP besides Young's modulus penalty function, and (2) design variables in TIMP are confined to the artificial material densities in SIMP. In order to demonstrate the validity and capability of TIMP, topology optimization models of minimizing weight with displacement/stress constraints under multiple loading cases are constructed. The unit virtual loading method is utilized to explicit displacement constraints, while the stress constraint globalization strategy is employed to convert enormous stress constraints into global structural distortion energy constraints. Three penalty functions in TIMP method play an important role to obtain sensitivities of constraints for free. The nonlinear programming algorithm is used for solutions, and the whole solution programming is implemented by Python scripts on ABAQUS software. Several numerical examples are presented for testing, and the effects of linear and nonlinear element weight penalty functions on the convergence speed are studied and discussed through numerical examples. It is demonstrated that, the proposed method is efficient and valid, and a nonlinear weight penalty function can yield higher convergence speed than a linear function.

2. Keywords: ICM method, SIMP method, TIMP method, displacement constraints, stress constraints

3. Introduction

The traditional structural topology optimization was early proposed by Maxwell at the end of the 19th century and further studied by Michell at the beginning of the 20th century. However, the modern structural topology optimization has been started since 1988, when Bendsøe and Kikuchi [1] proposed the Homogenization method (HM). The concept of continuum structural topology optimization was presented after that, as well as corresponding numerical methods. With the development of high performance computing science and technology, the research on numerical approaches for the continuum structural topology optimization has been made great progresses, and most of them are based on the "ground structure approach" [2]. Besides HM, the ground structure approach is represented with the Solid Isotropic Material with Penalization (SIMP) method [3], the Evolutionary Structural Optimization (ESO) method [4], the Independent Continuum and Mapping (ICM) method [5], the Level Set Method (LSM) [6] and so on.

Among these numerical approaches, the SIMP method is popular and has many practical applications because of its easy implementation. However, the development of SIMP method has stopped in a theory system with only one penalty function, which is the Young's modulus penalty function. While we thought about whether there was something we could do for its improvement, it's found out that the ICM method could be instructive. There are two reasons to do so. Firstly, the ICM method has made significant progresses over the decades. Its theory foundation is tamped and its modeling and solution approaches are tempered. Several numerical laws have been concluded within this method. Secondly, although the filter function in the ICM method has different definitions from the penalty function in the SIMP method, their mathematical formulations are similar. Therefore, it is possible to transplant progresses and ideas of the ICM method into the SIMP method. The transplanting work could achieve big developments of SIMP.

In SIMP method, the artificial relative density variables are defined between 0 and 1. The penalty function is formulated to put penalization on the Young's modulus of element with intermediate densities. The material used for an element yields to 0 or 1 by the penalization. Therefore, the core idea of SIMP is the concept of penalization. ICM method uses the independent continuous topological variables. The polish and filter functions are formulated to realize the higher-order approximations of the step function and its inverse function separately. The independent topological variables, which are discrete as 0 or 1 in nature, are mapped as continuous variables in [0, 1]. The continuous topological variables will be inverted into discrete variables at the end of optimization. The core idea of ICM is the concept of approximation.

If the penalization of SIMP is analogized with the approximation of ICM, new penalty functions can be presented: element weight penalty function and allowable stress penalty function. In previous application of SIMP, elemental weight is a linear function of the artificial relative densities. However, in ICM, several filter functions, including the element weight filter function, are nonlinear functions. It implies that the element weight penalty function could be linear and nonlinear. Therefore, this paper proposes a method of transplanting ICM ideas into material with penalization for the continuum structural topology optimization, which is called TIMP. TI represents Transplanting ICM Ideas, and MP is the latter half part of Solid Isotropic Material with Penalization. Since ICM and SIMP methods have not been compared to each other on the perspective of "ideas" before, the proposed TIMP method will further the developments of SIMP, and solve topology optimization problems easier than the traditional SIMP method.

The validity and capability of TIMP method is going to be demonstrated through three concrete tasks in this paper: (1) to formulate optimization model of minimizing weight with displacement constraints under multiple loading cases by TIMP, and (2) to construct optimization model of minimizing weight with stress constraints under multiple loading cases by TIMP and convert enormous local stress constraints into global structural distortion energy constraint by utilizing the stress constraint globalization strategy; (3)to provide unified formulations and solutions for the two optimization models, and to develop the solution process into secondary development software by Python scripts in ABAQUS.

4. TIMP method

In SIMP method, the Young's modulus penalty function is described as below,

$$E_i = \rho_i^{p_E} E_i^0 \tag{1}$$

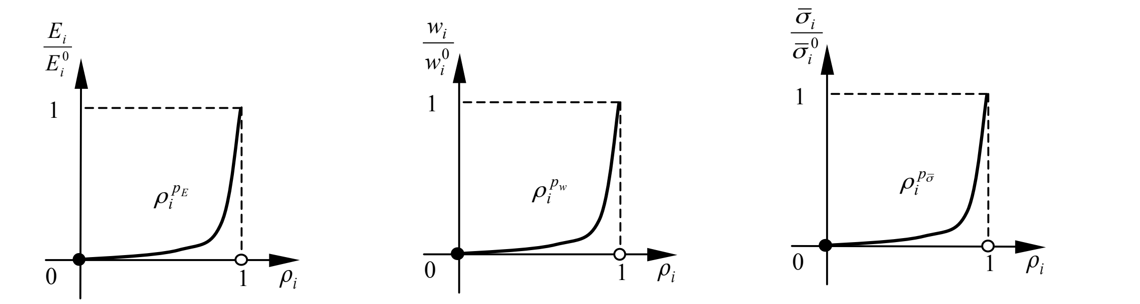
where the subscript "*i*" is the number of the element, and ρ_i denotes the element relative density variable (the ratio of the actual material density to the artificial material density). p_E is called the penalty factor. E_i^0 and E_i are the element Young's modulus for actual material and for artificial material separately.

With the idea of approximation in ICM method, different formulations of highly nonlinear and derivative filter functions are used to approximate inverse functions of the step functions [7]. Element weight, stiffness and allowable stress et al. are identified by them. This paper uses the method of analogy in the way that, the relative density variables and the penalty function in SIMP are, respectively, in analogy to the independent topology variables and the filter functions in ICM. Therefore, ideas of ICM method are transplanted into SIMP method, and two penalty functions in the form of power function similar to Eq.(1) are introduced as bellow,

$$w_i = \rho_i^{p_w} w_i^0 \tag{2}$$

$$\bar{\sigma}_i = \rho_i^{p_{\bar{\sigma}}} \bar{\sigma}_i^0 \tag{3}$$

where w_i and $\bar{\sigma}_i$ represent, respectively, the element weight and material allowable stress for elements with intermediate densities. w_i^0 and $\bar{\sigma}_i^0$ are, respectively, the initial element weight and initial material allowable stress for elements filled with actual material. p_w and $p_{\bar{\sigma}}$ are, respectively, the penalty factors of element weight and allowable stress. Obviously, Eq. (2) is linear when $p_w = 1.0$, and nonlinear when $p_w > 1.0$.



a) Young's modulus penalty function b) Weight penalty function c) Allowable stress penalty function

Figure 1: Curves for penalty functions in TIMP method

Based on the above analogies between ICM and SIMP methods, a new theory system on the basis of Eq. (1), Eq. (2) and Eq. (3) is proposed and called TIMP method. Their curves are plotted in Figure 1, where a) for the Young's modulus penalty function, b) for the element weight penalty function, and c) for the allowable stress penalty function.

Actually, Eq. (2) was usually used in a linear expression for researches by using SIMP method, but never presented in the name of penalty function. However, the TIMP method could have both linear and nonlinear expressions of the element weight penalty function. It is an expansion and development of the SIMP method.

5. Topology optimization problems with displacement constraints under multiple loading cases

The minimum-weight formulation for topology optimization with displacement constraints under multiple loading cases is usually expressed as below,

$$\begin{cases} \text{For} \quad \boldsymbol{\rho} \\ \text{Min} \;\; W(\boldsymbol{\rho}) \\ \text{s.t.} \quad u_{lr}(\boldsymbol{\rho}) \le \bar{u}_r \quad (l = 1,\cdots,L;\; r = 1,\cdots,R) \\ \qquad \rho_{\min} \le \rho_i \le 1 \quad (i = 1,\cdots,N) \end{cases} \tag{4}$$

where $\boldsymbol{\rho}$ is the artificial relative density vector, and $W(\boldsymbol{\rho})$ is the total weight of the structure. "l" and "r" are, respectively, the number of loading case and number of point with displacement constraint in a loading case. $u_{lr}(\boldsymbol{\rho})$ represents the displacement function at the r-th point of interest in the l-th loading case, and $\bar{u}_r$ is the allowable displacement at this point. "L" and "R" are, respectively, the total number of loading cases and total number of points with displacement constraints. "N" is the total number of elements. In order to avoid the singularity of the stiffness matrix, the minimum value of the artificial relative density is $\rho_{\min} = 0.001$.

Based on the TIMP method, element weights under different densities can be identified with Eq. (2), and the total weight of the structure can be expressed as below,

$$W(\boldsymbol{\rho}) = \sum_{i=1}^{N} w_i^0 \rho_i^{p_w} \tag{5}$$

According to the derivations in the paper [8], the displacement function was expressed explicitly with Eq. (1) by using the unit virtual loading method. Thus, the explicit expression for the displacement at the r-th point of interest in the l-th loading case can be stated as

$$u_{lr} = \sum_{i=1}^{N} u_{lri} = \sum_{i=1}^{N} \frac{D_{lri}^0}{\rho_i^{p_E}} \tag{6}$$

where u_{lri} represents the displacement contribution of the i-th element to the displacement at the r-th point of interest in the l-th loading case. D_{lri}^0 is the constant coefficient in the displacement contribution function of the i-th element to the displacement at the r-th point of interest in the l-th loading case.

6. Topology optimization problems with stress constraints under multiple loading cases

Topology optimization problems with stress constraints under multiple loading cases can be formulated as below,

$$\begin{cases} \text{For} \quad \boldsymbol{\rho} \\ \text{Min} \;\; W(\boldsymbol{\rho}) \\ \text{s.t.} \quad \sigma_{li} \le \bar{\sigma} \quad (l = 1,\cdots,L) \\ \qquad \rho_{\min} \le \rho_i \le 1 \quad (i = 1,\cdots,N) \end{cases} \tag{7}$$

where σ_{li} is the von Mises stress of the i-th element in the l-th loading case, and $\bar{\sigma}$ is the allowable stress. The objective can be expressed in the same way as Eq. (5).

The stress constraints globalization strategy is based on the von Mises yield criterion, and deals with the local stress constraints into a single combined relationship. This globalization strategy has been applied effectively in ICM method [9]. Its main ideas can be transplanted and used in TIMP method too. According to the von Mises yield criterion, when the element distortion energy density is no less than a certain allowable value, the strength of the material is about to yield. On the contrary, it is safe only if a relationship exists as below,

$$U_{li}^d = \frac{1+\nu}{3E_i}(\sigma_{\mathrm{VM}li})^2 \le \bar{U}_i^d = \frac{1+\nu}{3E_i}(\sigma_{\mathrm{Y}})^2 \tag{8}$$

where U_{li}^d is structural distortion energy density of the i-th element in the l-th loading case, and $\bar{U}_i^d$ is allowable distortion energy density of the i-th element. ν denotes the Poisson's ratio. $\sigma_{\rm Y}$ is the yield stress of the material (allowable stress), and $\sigma_{{\rm VM}li}$ represents the equivalent von Mises stress of the i-th element in the l-th loading case. It should be noted that a safety factor is required in Eq. (8) for practical engineering problems.

By multiplying the element volume V_i on both sides of Eq. (8) and performing the summation of all elements, the structural total distortion energy constraints can be obtained and described as below,

$$\sum_{i=1}^{N} U_{li}^{d} V_i \le \sum_{i=1}^{N} \overline{U}_i^{d} V_i \tag{9}$$

Thus, stress constraints are globalized into structural distortion energy constraints in Eq. (9), whose left side could be identified with Eq. (1) and right side could be identified with both Eq. (1) and Eq. (3). Their formulations are expressed as below,

$$\begin{aligned} V_i U_{li}^{d} &= \frac{(\rho_i^{(n-1)})^{p_E}}{\rho_i^{p_E}} (V_i U_{li}^{d})^{(n-1)} \\ V_i \overline{U}_i^{d} &= \frac{1+\nu}{3E_i^0 \rho_i^{p_E}} V_i (\overline{\sigma}_i^0 \rho_i^{p_{\overline{\sigma}}})^2 = V_i \overline{U}_i^{d0} \rho_i^{2p_{\overline{\sigma}} - p_E} \end{aligned} \tag{10}$$

where "n" is the number of iteration. $\rho_i^{(n-1)}$ is the relative densities obtained from the (n-1)-th iteration, and $(V_i U_{li}^{d})^{(n-1)}$ represents the distortion energy of the i-th element in the l-th loading case obtained from the (n-1)-th iteration, which can be computed by finite element analysis. $\overline{U}_i^{d0} = \frac{1+\nu}{3E_i^0} \overline{\sigma}_i^0$, and it represents the allowable distortion density for the element with solid material.

Eq. (9) could be also expressed in a different form as below,

$$\sum_{i=1}^{N} \frac{U_{li}^{d}}{\overline{U}_i^{d}} \le N \tag{11}$$

One should be noticed that $\sigma_{{\rm VM}li} \le \sigma_{{\rm Y}i}$ is a sufficient and unnecessary condition for Eq. (11). Therefore, the stress constraints in Eq. (7) can be replaced by the structural distortion energy constraints as below,

$$\sum_{i=1}^{N} \frac{(U_{li}^{d})^{(n-1)} (\rho_i^{(n-1)})^{p_E}}{\overline{U}_i^{d0} \rho_i^{2p_{\overline{\sigma}}}} \le N \zeta_l \tag{12}$$

where ζ_l is the adjusting factor of the structural distortion energy for the l-th loading case, and $\zeta_l = (\frac{\overline{\sigma}}{\sigma_{\rm VMmax}})^{\theta}$, where $\sigma_{\rm VMmax}$ denotes the maximum von Mises stress and θ is a constant determined by tests.

7. Unified models and solutions

In order to reduce the unnecessary calculation caused by the inactive constraints, whose left side values are far less than the right side values, only the active constraints can be selected to construct the optimization models. Thus, the subscripts l and r in displacement /stress constraints are merged into a single sequential number j, and the total number of active constraints is denoted by L_a.

Assuming that $x_i = \rho_i^{-p_E}$, $\underline{x}_i = 1$, and $\overline{x}_i = 0.001^{-p_E}$, a unified formulation for Eq. (4) and Eq. (7) with explicit objective and constraints formulations can be described as below,

$$\begin{cases} \text{For } x_i \quad (i = 1, \cdots, N) \\ \text{Min } W = \sum_{i=1}^{N} w_i^0 x_i^{-\eta} \\ \text{s.t. } \sum_{i=1}^{N} A_{ji} x_i^{\mu} - B_j \le 0 \quad (j = 1, \cdots, L_a) \\ \underline{x}_i \le x_i \le \overline{x}_i \end{cases} \tag{13}$$

For Eq. (4), $\eta = \frac{p_w}{p_E}$, $A_{ji} = D_{ji}^0$, $\mu = 1$, and $B_j = \overline{u}_j$. For Eq. (7), $\eta = \frac{p_w}{p_E}$, $A_{ji} = \frac{(U_{li}^{d})^{(n-1)} (\rho_i^{(n-1)})^{p_E}}{\overline{U}_i^{d0}}$,

$\mu = \dfrac{2p_{\bar{\sigma}}}{p_E}$, and $B_j = N\zeta_j$. Therefore, the sensitivities of objective and constraints are obtained for free by using methods and approached in Section 5 and Section 6.

The nonlinear programming algorithm, Dual Mapping Sequential Quadratic Programming (DMSQP), can be used for solutions of Eq. (13). Since the dual problem of Eq. (13) is formulated based on the dual theory, the number of design variables are reduced incredibly. Then, Sequential Quadratic Programming (SQP) is employed to address the dual problem based on its Kuhn-Tucker conditions. The whole solution programming is implemented by Python scripts on ABAQUS software, and results will be output automatically.

8. Numerical examples

In order to demonstrate the validity and capability of the TIMP method, a plate structure is studies here, and the influences of the linear element weight penalty function (LEWPF) and nonlinear element weight penalty function (NEWPF) on the convergence speed are observed specifically.

Figure2 shows the dimensions of a rectangular plate. The concentrated loading is $P_1 = P_2$ =3600N. Multiple loading cases are considered, and they are: Case 1 is to apply P_1 at the intersection of 1/3 horizontal and 1/2 vertical, Case 2 is to apply P_2 at the intersection of 2/3 horizontal length and 1/2 vertical, and Case 3 is to apply both P_1 and P_2 at the same time. The material properties are that, the Young's modulus E=210GPa, Poisson's ratio v=0.3, and density ρ =7800kg/m^3. The allowable stress of the material is 100MPa. Filtering schemes are utilized to alleviate the mesh-dependency and checker-board issues.

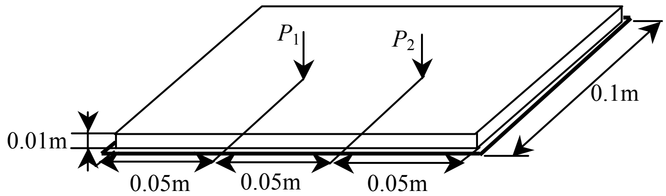


Figure 2: A rectangular plate with multiple loading cases

Firstly, topology optimizations with displacement constraints for the plate under different boundary conditions (BCs) are observed. It is required that the displacements along the loading direction at the loading points are no more than 0.028mm when the plate is clamped, and no more than 0.064mm when it is simply-supported. Both of LEWPF (p_w =1.0) and LEWPF (p_w >1.0) are used and optimization parameters are set by trial and errors. Figure 3 and Figure 4 show the optimum topologies of the clamped and simply-supported plates separately. The parameters and final results are presented in Table 1. It is found out that, the optimum topologies are similar while under the same BCs, and the utility of NEWPF yield less iterations and better satisfaction with stress constraints than the utility of LEWPF.

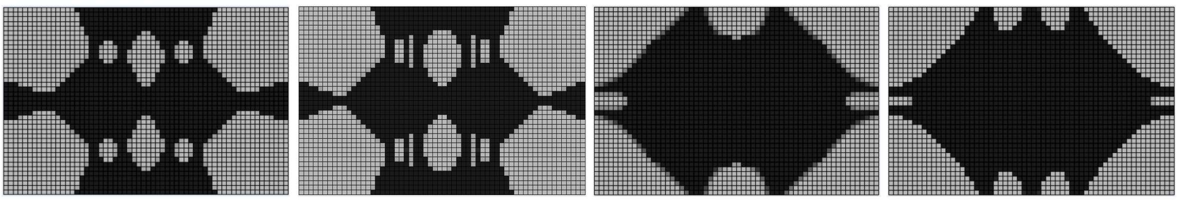
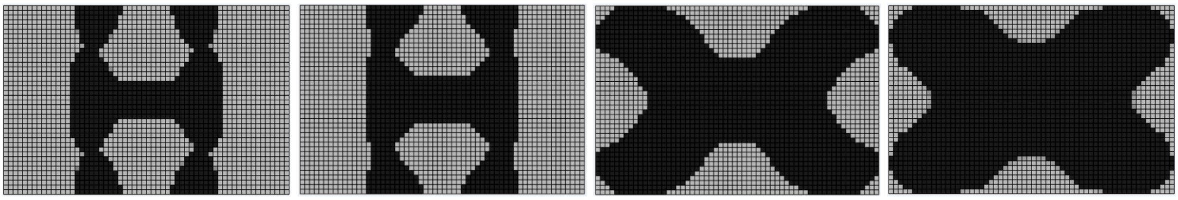


Figure 3: Topologies under displacement constraints: a) clamped plate with LEWPF, b) clamped plate with NEWPF, c) simply-supported plate with LEWPF, and d) simply-supported plate with NEWPF

Table 1: Optimization parameters and results for clamped plate with displacement constraints

BCs.	p_w	p_E	Iterations	Displacement / mm			Weight reduced by / %
				Case 1	Case 2	Case 3	
Clamped	1.0	3.0	30	0.023	0.023	[0.028, 0.028]	51.28
	1.5	4.5	20	0.024	0.024	[0.028, 0.028]	52.99
Simply-supported	1.0	3.0	36	0.042	0.043	[0.064, 0.064]	39.32
	1.5	4.5	31	0.042	0.042	[0.062, 0.062]	40.17

Secondly, topology optimizations with stress constraints for the plate under different BCs are studied too. Figure 5 and Figure 6 show the optimum topologies of the clamped and simply-supported plates separately. The parameters and final results are presented in Table 2.



e) f) g) h)

Figure 4: Topologies under stress constraints: e) clamped plate with LEWPF, f) clamped plate with NEWPF, g) simply-supported plate with LEWPF, and h) simply-supported plate with NEWPF

Table 2: Optimization parameters and results for clamped plate with displacement constraints

BCs.	p_w	p_E	$p_{\bar{\sigma}}$	θ	Iterations	Max von Mises Stress / MPa			Weight reduced by / %
						Case 1	Case 2	Case 3	
Clamped	1.0	3.5	1.7	0.5	26	97.34	97.34	103.68	68.12
	1.5	3.5	2.0	1.5	23	85.10	85.10	92.77	65.47
Simply-supported	1.0	3.0	2.0	3.0	13	78.52	78.52	101.86	30.78
	1.5	3.5	2.0	1.5	11	76.36	76.36	90.42	21.07

9. Conclusions

The proposed TIMP method is a development of the SIMP method, and its application in addressing topology optimization models with displacement and stress constrains under multiple loading cases are detailed discussed in this paper. Numerical examples of a rectangular plate under different boundary conditions demonstrate that using nonlinear element weight penalty function yields better results and higher convergence speed than using linear element weight penalty function. It is indicated that the TIMP method is valid and capable to address complicated topology optimization problems.

10. Acknowledgements

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