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# Structural approximations for composite optimisation

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## 1 Abstract
A way to approximate the response of, variable stiffness, composites for optimisation is explained. A two-level approximation scheme is proposed inspired by traditional approximation concepts such as force approximations. First an approximation in terms of the in- and out-of-plane stiffness matrices is made. In the stiffness approximation a generalised convex linearisation approach is used: compliance, stress and inverse buckling load are approximated in terms of the linear and reciprocal stiffness matrices. Either the lamination parameters, or the nodal fibre angle distribution are used as design variables. A quadratic approximation is used to build the approximations in fibre angle space. Conservativeness is guaranteed by adding a convex damping function to the approximations. The method of conservative, convex separable approximations is used for the optimisation.

**2 Keywords:** Optimisation, variable stiffness, approximations, multi-level

## 3 Introduction
Composite materials are attractive due to their high stiffness- and strength-to-weight ratio. It has been shown that by spatially varying the stiffness of the composite, better performance can be obtained without adding extra weight. To optimise variable stiffness laminates, a three-step optimisation approach has been developed. [1, 2] In step one the optimal stiffness distribution is found, in step two the optimal fibre angles are obtained, and in step three the optimal fibre paths are retrieved. The critical numerical parts of this optimisation are the approximations used.

Response surfaces can be used as an approximation method to reduce the computational effort. This method is accurate, but choosing the response surfaces is problem-dependent. [3] The Rayleigh-Ritz method is often used to approximate eigenvalues, or buckling factors. It has been used to optimise buckling factors, but the shape functions need to be chosen, which is also problem-dependent. [4]

A generally applicable method is linearisation. Since stress and displacement are linear functions of the reciprocal sizing variables in a statically determinate structure, they are often reciprocally approximated. The 'convex linearisation' (ConLin) method introduced by Fleury and Braibant is a generalisation of linear or reciprocal approximations [5]. Whether a variable is approximated linearly or reciprocally depends on the derivative at the approximating point: linear if the derivative is positive, reciprocal if the derivative is negative. Vanderplaats recognised that approximating structural responses using the force in a member led to better approximations, called force approximations. [6] A two-level approximation is formulated. On level one, the stress is approximated as a function of the section properties. On level two, the section properties are approximated in terms of the physical properties. Once level two has converged, the stress is calculated based on the new properties, and a new level one approximation is made. This is repeated until the stress converges. [6, 7]

In this paper the ideas of force approximations and the ConLin method are combined. In level one approximations the structural responses are approximated in terms of the stiffness matrices, using a generalised version of the ConLin approximations. For level two approximation, a quadratic *approximation of approximation* is constructed along the lines of the Gauss-Newton method.

This paper is organised as follows: the approximations in terms of the stiffness are given in section 4. The lamination parameter and fibre angle approximation are discussed in sections 5 and 6. The way these approximations are used is explained in section 7 and this paper is concluded in in section 8.

## 4 Approximations in stiffness space
The level one approximation, in stiffness space, is derived for three different structural responses. Compliance is discussed first since, as will be shown, it is the most straight-forward. Next, stress and buckling are discussed.

### 4.1 Compliance approximations
To clarify the rationale of the approximation, the derivation is first done for elastic trusses, then for general structures.

### 4.1.1 Elastic trusses

The strain energy of a truss made of an elastic material, with only $A_e$, the area of each element, as design variables is defined as:

$$U = \sum_e \frac{1}{2} \cdot E \cdot \varepsilon_e^2 \cdot A_e \cdot l_e \tag{1}$$

where E is the Young's modulus, $l_e$ is the length and $\varepsilon_e$ is the strain of element $e$. Using the standard strain-displacement equation, the principle of minimal total potential energy can be written as

$$\min_{\varepsilon,u} \sum_e \frac{1}{2} \cdot E \cdot \varepsilon_e^2 \cdot A_e \cdot l_e - f^t \cdot u \quad \text{s.t.} \quad \varepsilon_e - {b_e}^T \cdot u = 0 \tag{2}$$

The Lagrangian of this function can be formulated. Taking the Lagrange dual, and using the Young-Fenchel-Moreau transform leads to

$$\max_{\sigma} \left( - \left( \sum_e f^* (\sigma_e) A_e l_e \right) + \min_u \left( \sum_e \sigma_e A_e l_e b_e - f \right)^T \cdot u \right) \tag{3}$$

$$f^* (\sigma_e) = \sigma_e \cdot \varepsilon_e - g(\varepsilon) \quad \text{and} \quad \sigma_e = \frac{\partial g(\varepsilon)}{\partial \varepsilon} \tag{4}$$

Using the standard equation for stress and assuming that strong duality holds, equation (2) can be written as:

$$\min_F \sum_e f^* \left( \frac{F_e}{A_e} \right) \cdot A_e \cdot l_e \quad \text{s.t.} \quad \sum_e F_e \cdot b_e \cdot l_e = f \tag{5}$$

Implementing the correct expression for $f^*$, the minimal total potential energy and minimal compliance can be written as

$$C^* = \min_A \left( \min_F U^* \right) = \min_A \left( \min_F \left( \frac{1}{2} \sum_e \frac{F_e^2 \cdot l_e}{E \cdot A_e} \right) \right) \tag{6}$$

The minimisation over the, equilibrated, element forces corresponds to structural analysis, the minimisation with respect to the element areas corresponds to structural design. Since this is a min-min formulation convergence is guaranteed by alternating analysis and design.

Using the force $F_e$ at iteration $k$ in equation 6, the compliance is approximated in terms of $A_e$. This approximation has four desirable properties:

1. separable: each element area $A_e$ only depends on properties of element e.
2. convex: the second derivative with respect to the element areas is larger than zero.
3. homogeneous: doubling $A_e$ halves the compliance, hence the approximation is homogeneous of order -1.
4. conservative: the internal force $F_e$ does not change as a function of the area, hence $F^{(k)}$ is always feasible, furthermore the compliance of $F^{(k+1)}$ is at least as good as the approximation based on $F^{(k)}$.

These four properties mean the optimisation using this approximation is numerically efficient (separable), a feasible solution can always be found (homogeneous), has a solution (convex) and is globally convergent (conservative). Ideally, a good structural approximation should retain as much as possible of these four properties.

### 4.1.2 General structures

Three equations need to be satisfied for a general structure:

1. equilibrium: $N_{\mu\nu,\nu} + b_\mu = 0$
2. strain-displacement: $\varepsilon^0_{\mu\nu} = \frac{1}{2} \left( u_{\mu,\nu} + u_{\nu,\mu} \right)$
3. material law: $N_{\mu\nu} = A_{\mu\nu\alpha\beta} \cdot \varepsilon_{\alpha\beta}$

where $N$ is the generalised stress, $\varepsilon^0$ is the strain at mid-plane, and $A$ is the generalised stiffness matrix.
Assuming the strain-displacement is satisfied in weak form, and denoting the test function as $\tilde{N}$:

$$\int_\Omega \tilde{N}_{\mu\nu} \cdot \varepsilon^0_{\mu\nu} - \frac{1}{2} \cdot \tilde{N}_{\mu\nu} \left( u_{\mu,\nu} + u_{\nu,\mu} \right) d\Omega = 0 \tag{7}$$

Taking the symmetry of the second term into account, noting that $\left( \tilde{N}_{\mu\nu} \cdot u_\mu \right)_{,\nu} = \tilde{N}_{\mu\nu} \cdot u_{\mu,\nu} + \tilde{N}_{\mu\nu,\nu} \cdot u_\mu$, and using the Gauss theorem, equation 7 becomes:

$$\int_\Omega \tilde{N}_{\mu\nu} \cdot \varepsilon^0_{\mu\nu} d\Omega + \int_\Omega \tilde{N}_{\mu\nu,\nu} \cdot u_\mu d\Omega - \oint_{\Gamma_t} \tilde{N}_{\mu\nu} \cdot u_\mu \cdot n_\nu dA = 0 \tag{8}$$

Defining $\tilde{N}_{\mu\nu}$ as a change in stress distribution that satisfies the equilibrium equation $\partial N_{\mu\nu}$, means the second term equals zero. Furthermore, the normal of $\partial N_{\mu\nu}$ is zero, thus the third term is zero as well. Hence

$$\int_{\Omega} \partial N_{\mu\nu} \cdot \varepsilon^0_{\mu\nu} d\Omega = 0 \tag{9}$$

Noting that $\varepsilon^0_{\mu\nu} = \frac{\partial U^*(N_{\mu\nu})}{\partial N_{\mu\nu}}$ and using the material law, the total strain energy and minimum compliance can be written as:

$$C^* = \min_A \min_N \int_{\Omega} U^*(N,A) d\Omega = \min_A \min_N \int_{\Omega} \frac{1}{2} \cdot A^{-1}_{\alpha\beta\mu\nu} N_{\mu\nu} N_{\alpha\beta} d\Omega \tag{10}$$

In the general case, the compliance is in terms of $A^{-1}$, thus the approximation should also be in this form. Defining the load and strain at mid-plane as:

$$N = \begin{bmatrix} N_x & N_y & N_{xy} \end{bmatrix}^T \qquad \varepsilon^0 = \begin{bmatrix} \varepsilon^0_x & \varepsilon^0_y & \gamma^0_{xy} \end{bmatrix}^T \tag{11}$$

the compliance can be written as:

$$U^* = \frac{1}{2} N^T \cdot A^{-1} \cdot N = \frac{1}{2} (N \cdot N^T) : A^{-1} = \phi : A^{-1} \tag{12}$$

where : denotes the Frobenius inner product, meaning $A : B = trace(A \cdot B^T)$

This approximation can be shown to be convex, in terms of $A$, the normalised stiffness matrix $\hat{A}$ and the height $h$. The approximation in equation (10) is also separable: the normalised general stiffness matrix and height do not influence other elements. Conservativeness also still holds: the minimisation in terms of $A$ is still done in a feasible space for $N$. Homogeneity of order $-1$ still holds, since the form of the approximation has not changed. Hence, the approximation still has the four desirable properties.

### 4.2 Stress approximations

Next, an approximation for stress is derived. This is often used since the (local) stress constraints are often a driving factor. The same approach as that for compliance is used: first develop approximations for elastic trusses, then general structures.

#### 4.2.1 Elastic trusses

Stress is usually set as constraint, and is expressed as a failure index, which also normalises the stress:

$$r_e \approx \frac{F_e^{(k)}}{\sigma_{all} \cdot A_e} \tag{13}$$

where $\sigma_{all}$ denotes the maximum allowable stress. However, a modification has to be added since $F_e$ is not just dependent on the area of element $e$, it is also dependent on the other areas. Since we are interested in creating a homogeneous approximation, this modification should have two properties: one, for the current area, it should be zero, two, if all areas are scaled with the same factor, it should remain zero. The total approximation becomes a linear-reciprocal function:

$$r_e \approx \frac{F_e^{(k)}}{\sigma_{all} \cdot A_e} + \sum_e a_e \cdot A_e \tag{14}$$

Hence, the approximation is separable in terms of the areas. The function is convex: the second derivative with respect to the element area is positive. homogeneity also holds in a limited sense: scaling the area will scale the stress approximation. The approximation is not necessarily conservative. How this lack of conservativeness is handled is explained in section 7.

#### 4.2.2 General structures

For a general structure, the failure index is a function of strain $r_e = r_e(\varepsilon_e)$ and may be locally approximated by [8]:

$$r = N^T A^{-1} g + s_i \cdot \Delta N \quad \text{with} \quad g^{(k)} = \left. \frac{\partial r}{\partial \varepsilon} \right|_{\varepsilon = \varepsilon^{(k)}} \tag{15}$$

Hence, the stress has both a local term, depending on $A^{-1}$, and a global term due to the load redistribution $\Delta N$. The local approximation of $r$ is given by

$$r = \phi : A^{-1} \quad \text{with} \quad \phi = \frac{1}{2}\left(Ng^T + gN^T\right) \tag{16}$$

Upon inspection, it was found $\phi$ is not positive semi-definite (psd), which is necessary for convexity. To guarantee convexity, the term $\phi$ is divided in a psd and a non-psd part. [8] The indefinite part is approximated using a linear expansion. The details are explained in Khani et. al. [8]

The load redistribution part is approximated in a linear way, similar to the force approximation introduced by Vanderplaats and Thomas [6]. The total approximation of the stress is linear-reciprocal:

$$r \approx \phi : A^{-1} + \psi : A \tag{17}$$

which is the same form as was found for trusses. This function is convex and homogeneous by construction, it is also separable even though the load redistribution is taken into account. The approximation, again, is not necessarily conservative.

### 4.3 Buckling optimisation

The buckling factor $\lambda$ is calculated using [2]

$$\left(K^b - \lambda K^g\right) \cdot a = 0 \tag{18}$$

where $K^b$ is the global bending stiffness matrix and $K^g$ is the global geometric stiffness matrix. $a$ is the mode shape normalised with respect to $K^b$. Deriving equation (18) with respect to a design variable $A$ leads to

$$\frac{d\lambda}{dA} = \lambda a^T \cdot \left(\frac{dK^b}{dA} + \lambda \frac{dK^g}{dA}\right) \cdot a \tag{19}$$

The first term of this equation is the derivative of the bending stiffness, which is local. The second term is the derivative of the geometric stiffness, which is not local: it represents the load redistribution due to the stiffness change of a single element. It can be seen the buckling factor is homogeneous of order zero with respect to the in-plane stiffness matrices from the general FE equilibrium and material law. Hence, $\lambda$ depends on the load redistribution. Furthermore, $\lambda$ is homogeneous of order one with respect to the bending stiffness. The inverse buckling load $r = 1/\lambda$ is approximated since it behaves similarily to the compliance and stress responses. This has the same homogeneity: order zero in terms of the in-plane stiffness and order one in terms of the *inverse* bending stiffness. The approximation has the following form: [2]

$$r \approx \sum_n \psi : A + \phi : D^{-1} \tag{20}$$

where the sum over all nodes is taken and no constant is added since the first part is zero at the approximation point, and the second part is exactly the inverse buckling load at the approximation point. It is shown in IJsselmuiden et al. [2] that the approximation is convex. Looking at equation (20), separability is also satisfied, and homogeneity is satisfied by construction. Just as for stress, conservativeness is not guaranteed.

## 5 Approximations in lamination parameter space

In step one of the three-step optimisation approach, the stiffness is optimised in terms of the lamination parameters. This is not really a level two approximation since the lamination parameters describe the stiffness matrices exactly:

$$A = h \cdot (\Gamma_0 + \Gamma_1 \cdot V_1 + \Gamma_2 \cdot V_2 + \Gamma_3 \cdot V_3 + \Gamma_4 \cdot V_4) \tag{21}$$

$$D = \frac{h^3}{12} \cdot (\Gamma_0 + \Gamma_1 \cdot W_1 + \Gamma_2 \cdot W_2 + \Gamma_3 \cdot W_3 + \Gamma_4 \cdot W_4) \tag{22}$$

where $h$ is the thickness of the laminate and $\Gamma$ are the laminate invariant matrices.

The feasible region of either the in- or out-of-plane lamination parameters separate is defined as

$$\begin{gathered} 2 \cdot V_1^2 \cdot (1 - V_3) + 2 \cdot V_2^2 \cdot (1 + V_3) + V_3^2 + V_4^2 - 4 \cdot V_1 \cdot V_2 \cdot V_4 \leq 1 \\ V_1^2 + V_2^2 \leq 1 \\ -1 \leq V_3 \leq 1 \end{gathered} \tag{23}$$

If balanced laminates are desired, $V_2$ and $V_4$ are set to zero. Note that although $V$ is used, this also holds for the out-of-plane lamination parameters $W$. For the combination of in- and out-of-plane lamination parameters there is no easy relationship. The feasible region has been defined in earlier work. [9, 10] The design space has been shown to be convex in terms of the lamination parameters. [1]

## 6 Approximations in fibre angle space

In step two of the three-step optimisation approach, the lay-up at the different nodes is optimised. Based on the approximation in stiffness space, a Taylor expansion in terms of the change in fibre angles is made:

$$f^{(2)}(\theta) \approx f_0^{(1)} + g \cdot \Delta\theta + \Delta\theta^T \cdot H \cdot \Delta\theta = f^{(1)}(A(\theta)) \tag{24}$$

where $f_0^{(1)}$ denotes the value, $g$ the gradient and $H$ is an approximation of the Hessian of the first approximation in stiffness space at the approximation point; $A$ contains the components of the stiffness matrices $A$ and $D$. $g_i$ can be found by deriving equation (24). Deriving a second time, the Hessian is found to be

$$H_{ij} = \frac{\partial^2 f^{(1)}}{\partial\theta_i \partial\theta_j} = \frac{\partial^2 f^{(1)}}{\partial s_\alpha \partial s_\beta} \cdot \frac{\partial s_\alpha}{\partial \theta_i} \cdot \frac{\partial s_\beta}{\partial \theta_j} + \underline{\frac{\partial f^{(1)}}{\partial s_\alpha} \cdot \frac{\partial^2 s_\alpha}{\partial\theta_i \partial\theta_j}} \tag{25}$$

Convexity is guaranteed by omitting the underlined part of equation (25), which is not guaranteed to be psd, and using only the Gauss-Newton part which is always psd. Since an approximation has to have equal function and gradient at the approximation point as the approximated function, using only part of the Hessian gives a valid approximation which is convex and separable. Note that homogeneity is not relevant in fibre-angle space and is not discussed.

## 7 Using the approximations

The method of conservative convex separable approximations (CCSA) is used. [11] As shown in the previous sections, the approximations, either in terms of the stiffness or the fibre angles, are always convex and separable. To assure the approximation is conservative, a damping function $d$ is added:

$$f \approx \hat{f}(x) + \rho \cdot d(x) \tag{26}$$

where $\hat{f}$ is an approximation, and $\rho$ is the damping factor. This was done for both levels of approximation: both for stiffness and fibre angle approximation a damping function was added. To make the notation clear: a 1 denotes the approximation in stiffness space a 2 indicates fibre angles, if no superscript is added, it holds for both approximations.

The functions need to satisfy the following conditions: [11]

- the functions are continuous and the gradient and Hessian with respect to x exist and are continuous
- at the approximation point $\hat{f}(x) = f(x)$ and $d(x) = 0$
- at the approximation point $\nabla_x \hat{f}(x) = \nabla_x f(x)$ and $\nabla_x d(x) = 0$
- $\nabla^2_{xx} \hat{f}(x)$ is positive semi-definite
- $\nabla^2_{xx} d(x)$ is positive definite

The solution procedure is explained in algorithm 1. This algorithm is used to optimise a curved plate for buckling, while constraining the compliance in another paper by the authors. [12]

**Algorithm 1** Solution Procedure

1: start from an initial fibre angle distribution.
2: perform an FEA and calculate the sensitivities for the first level approximation $f^{(1)}$.
3: add the damping function $\rho^{(1)} \cdot d^{(1)}$ to the first level approximation.
4: calculate the gradient and Hessian for the second level approximation $f^{(2)}$.
5: add the damping function $\rho^{(2)} \cdot d^{(2)}$ to the second level approximation.
6: apply the steering constraint, build the Lagrangian $\mathscr{L}$ and solve the system.
7: calculate first level approximation $f^{(1)}$ and update damping factor of level two $d^{(2)}$.
8: if the first level approximation $f^{(1)}$ is improved, continue, else go back to step 5.
9: if the first level approximation $f^{(1)}$ has converged, or the maximum number of iterations is reached, continue, else return to step 4.
10: perform an FEA and update the damping factor of first level approximation $d^{(1)}$.
11: if the FE response has improved, continue, else go back to step 3.
12: if FEA has converged, the optimal fibre angle distribution is found, else return to step 2.

## 8 Conclusion

An approximation approach for the optimisation of, variable stiffness, composites is proposed. The three-step optimisation approach for variable stiffness composites is used as guideline. In step one the optimal stiffness distribution is found, in step two the optimal nodal fibre angle distribution is found, in step three the fibre paths are found. The approximations in this paper focus on step one and two.

Analogous to the force approximation a two-level approximation approach is proposed. In level one, a generalised ConLin method is used: the structural responses are approximated in terms of the in- and out-of-plane stiffness matrices and their inverse. In step one, lamination parameters are used to optimise the stiffness distribution. Since the stiffness matrices are perfectly described by the lamination parameters, this is not a level two approximation. In step two, a second-order Taylor approximation in terms of the change in fibre angles is made at the approximation point as a level two approximation.

The conservative, convex separable approximation method is used during the optimisation. Convexity and separability are guaranteed by construction of the approximations. Conservativeness is not guaranteed. A damping function and damping factor are added to the stiffness and fibre angle approximations to guarantee conservativeness and thus global convergence.

In this paper only compliance, stress and inverse buckling load have been approximated, but the method is more generally applicable. Other responses may be approximated in a similar manner.

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## References

[1] IJsselmuiden, S. T., *Optimal design of variable stffness composite structures using lamination parameters*, Ph.D. thesis, Delft University of Technology, 2011.

[2] Ijsselmuiden, S. T., Abdalla, M. M., and Gürdal, Z., "Optimization of Variable-Stiffness Panels for Maximum Buckling Load Using Lamination Parameters," *AIAA Journal*, Vol. 48, No. 1, 2010, pp. 134–143.

[3] Venter, G., Haftka, R. T., and Starnes, J. H., "Construction of Response Surface Approximations for Design Optimization," *AIAA Journal*, Vol. 36, No. 12, 2015/02/17 1998, pp. 2242–2249.

[4] Wu, Z., Weaver, P. M., Raju, G., and Kim, B. C., "Buckling analysis and optimisation of variable angle tow composite plates," *Thin-Walled Structures*, Vol. 60, No. 0, 2012, pp. 163 – 172.

[5] Fleury, C., "CONLIN: An efficient dual optimizer based on convex approximation concepts," *Structural optimization*, Vol. 1, No. 2, 1989, pp. 81–89.

[6] Vanderplaats, G. and Thomas, H., "An improved approximation for stress constraints in plate structures," *Structural optimization*, Vol. 6, No. 1, 1993, pp. 1–6.

[7] Vanderplaats, G., Kodiyalam, S., and Long, M., *A two-level approximation method for stress constraints in structural optimization*, American Institute of Aeronautics and Astronautics, 2015/02/17 1989.

[8] Khani, A., IJsselmuiden, S., Abdalla, M., and Gürdal, Z., "Design of variable stiffness panels for maximum strength using lamination parameters," *Composites Part B: Engineering*, Vol. 42, No. 3, 2011, pp. 546 – 552.

[9] Setoodeh, S., Abdalla, M., and Gürdal, Z., *Approximate Feasible Regions for Lamination Parameters*, American Institute of Aeronautics and Astronautics, 2015/02/18 2006.

[10] Wu, Z., Raju, G., and Weaver, P., *Feasible Region of Lamination Parameters for optimization of Variable Angle Tow (VAT) Composite Plates*, American Institute of Aeronautics and Astronautics, 2015/02/18 2013.

[11] Svanberg, K., "a class of globally convergent optimization methods based on conservative convex separable approximations," *Siam J. optim*, Vol. 2, 2002, pp. 555–573.

[12] Peeters, D., Hesse, S., and Abdalla, M., "Stacking Sequence Optimisation of Variable Stiffness Laminates with Manufacturing Constraints," *Composites Structures*, accepted.