Topology optimization of plate structures subject to initial excitations for minimum dynamic performance index

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1. Abstract
This paper studies optimal topology design of damped vibrating plate structures subject to initial excitation. The design objective is to minimize an integrated square performance measure, which is often used in optimal control theory. The artificial density of the plate element is the topology design variable. The Lyapunov's second method is applied to reduce the calculation of performance measure to the solution of the Lyapunov equation. An adjoint variable method is developed in our study, which only needs to solve the Lyapunov equation twice. However, when the problem has a large number of degrees of freedom, the solution process of Lyapunov equation is computational costly. Thus, the full model is transform to a reduced space by mode reduction method. And we propose a selection method to decrease the number of eigenmodes to further reduce the scale of reduced model. Numerical example of optimum topology design of bending plates is presented for illustrating validity and efficiency of our new algorithm.

2. Keywords: Adjoint method, vibration control, topology optimization

3. Introduction
Structural topology optimization and structural vibration control have called attention both in theoretical research and practical applications in engineering. Structural topology optimization provides a powerful automated tool for improving the structural performance in the initial conceptual design stage. Usually, optimization problems are formulated to minimize the material usage or to optimize the structural performance. Structural vibration control is a particularly important consideration in dynamic system design. Many control algorithms have been developed for passive and active control. Passive control systems that do not require any external power are widely used to reduce the response of structures.

In engineering applications, shell structures are widely used. Structural topology optimization and structural vibration control of shell structures has received an ever increasing attention. Several researchers have applied structural topology optimization techniques to structural vibration control problems. In most of existing works, structural topology optimization techniques are used to obtain the layout of piezoelectric or damping material on a main structure. Kang et al. investigate the optimal distribution of damping material in vibrating structures subject to harmonic excitations by using topology optimization method [1]. Chia et al. introduced cellular automata algorithms into the layout optimization of damping layers [2]. Zheng et al. dealt with topology optimization of plates with constrained layer damping treatment for maximizing the sum of the modal damping ratios, which are approximated with the modal strain energy method [3]. In this paper, the problem of a plate or shell just contains damping material will be considered.

Many performance indices have been considered in vibration control optimization problems, like H_2 or H_∞ norms. In time domain, there is a classic problem formulation of passive structural vibration control that deals with the dynamic system disturbed by initial conditions. The objective is to find design parameters of the damped vibration system that minimize the performance index in the form of time integral of the quadratic function of state variables (displacement and velocities, e.g. see equation (5)). This performance index can be evaluated by Lyapunov's second method [4]. Based on the Lyapunov equation, the evaluation of performance indices are simplified into matrix quadratic forms and do not require the time domain integration. Parameter optimization problems with a quadratic performance index have been solved by this method [5]. Wang et al. applied the Lyapunov equation to solve the transient response optimization problem of linear vibrating systems excited by initial conditions [6]. Du applied the Lyapunov equation to obtain the optimum configuration of dynamic vibration absorber (i.e., DVA) attached to an undamped or damped primary structure [7].

A well-known efficient solution technique for calculating the dynamic response of structures is to transform the model into a reduced space. Various methods for this requirement are available now, such as the Guyan reduction, mode superposition, modal acceleration and Ritz vector methods [8]. Among others, the mode superposition method is generally recognized as an efficient approach for dealing with large-scale proportionally damped structures. Generally, the structural response of reduced model is expressed as a linear combination of their first dozens or hundreds eigenmodes. However, for some cases, the eigenmodes of low order may have no effect on the structural response of reduced model. In this paper, a selection method is used to find these eigenmodes to decrease the number of basis vectors to further reduce the scale of the reduced model.

In this paper, an approach is developed for topology optimization involving a quadratic performance index of linear elastic shell structure subject to initial excitations. Mode reduction method and eigenmode selection method are used to decrease the computing time of optimization process. At last, a cantilever plate example and several illustrative results are presented.

4. Topology optimization problem formulation

4.1 Governing equations

Consider a viscously damped linear vibration system governed by the equation:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{1}$$

where $\mathbf{M}$($N\times N$) is the mass matrix, $\mathbf{C}$($N\times N$) is the damping matrix, $\mathbf{K}$($N\times N$) is the stiffness matrix, and $\mathbf{u}$($N\times 1$) is displacement vector. N is the structural degree of freedoms. Assume the system is excited by initial displacements or velocities. And the design problem is to find in $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ matrices to minimize a performance matrix in the form

$$J = \int_0^T q(\mathbf{u}, \dot{\mathbf{u}})\,\mathrm{d}t \tag{2}$$

where, $q(u, \dot{u}) = \mathbf{u}^{\mathrm{T}}\mathbf{Q}_u\mathbf{u} + \dot{\mathbf{u}}^{\mathrm{T}}\mathbf{Q}_{\dot{u}}\dot{\mathbf{u}}$ is a quadratic function of $\mathbf{u}$ and $\dot{\mathbf{u}}$. Transient dynamic responses have to be performed to evaluate the objective function. Direct or adjoint methods can be applied to evaluate the response sensitivity required for evaluation sensitivity of the performance. Alternative, if we replace the upper bound of integration to infinite, we can use Lyapunov's second method to evaluate the performance without performing transient dynamic response analysis.

To apply Lyapunov's second method to this system, it is necessary to rewrite Eq.(1) in the state space form

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} \tag{3}$$

Where

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix} \tag{4}$$

The matrix $\mathbf{A}$ is ($2N\times 2N$). The vector $\mathbf{X}$ is ($2N\times 1$). Structural design parameters such as mass density, damping ratio and spring stiffness are contained in the matrix $\mathbf{A}$. The optimization problem is to choose these parameters to minimize the performance measure $\boldsymbol{J}$ defined by

$$J = \int_0^\infty \mathbf{X}^{\mathrm{T}}\mathbf{Q}\mathbf{X}\,\mathrm{d}t \tag{5}$$

for a given initial excitation $\mathbf{X}(0)$. In Eq.(5), $\mathbf{Q}$($2N\times 2N$) is a positive semi-definite symmetric weighting matrix and t denotes time. According to Lyapunov theory of stability, for an asymptotically stable system, there exist a symmetric positive semi-definite matrix $\mathbf{P}$($2N\times 2N$) satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{6}$$

Eq.(6) is the well-known Lyapunov equation. Based on the Lyapunov's second equation, the Eq.(5) can be further simplified as

$$J = \mathbf{X}(0)^{\mathrm{T}}\mathbf{P}\mathbf{X}(0) \tag{7}$$

That is to say, to minimize $\boldsymbol{J}$ in Eq.(5) is equivalent to minimize $\mathbf{X}(0)^{\mathrm{T}}\mathbf{P}\mathbf{X}(0)$, where $\mathbf{X}(0)$is the initial state vector and the unknown symmetric matrix $\mathbf{P}$ can be obtained by solving Eq.(6).

4.2 Mathematical formulation of topology optimization problem

In this paper, the topology optimization problem for finding the optimal distribution of given material to minimize the quadratic integral form structural performance index of a vibrating structure excited by initial excitation is considered. The mathematical formulation of topology optimization problem is expressed as

$$\begin{aligned} &\mathit{find} \ \ (\rho_1, \rho_2, \ldots, \rho_{N_e}) \\ &\min \ \ J = \int_0^\infty \mathbf{X}^{\mathrm{T}}\mathbf{Q}\mathbf{X}dt \\ &s.t. \ \ \dot{\mathbf{X}} = \mathbf{A}\mathbf{X} \end{aligned}$$

$$\sum_{i=1}^{N_e} \rho_i V_i - V_{frac} \times \sum_{i=1}^{N_e} V_i^e = 0$$

$$0 \le \rho_{\min} \le \rho_i \le 1,\ i = 1, 2, \ldots, N_e \tag{8}$$

where, ρ_i is the artificial density of ith element, $\rho_{\min}$ is lower bound of artificial density, V_i^e is the volume of ith element, V_{frac} is the specific volume fraction, and N_e is the number of elements in design domain.

An artificial damping material model that has a similar form as the SIMP approach is used and the artificial densities of elements are taken as design variables. The elemental mass matrix and stiffness matrix are expressed by

$$\mathbf{M}_i = \rho_i \widetilde{\mathbf{M}}_i,\ \mathbf{K}_i = \left[\frac{\rho_{\min} - \rho_{\min}^p}{1 - \rho_{\min}^p} \left(1 - \rho_i^p\right) + \rho_i^p \right] \widetilde{\mathbf{K}}_i \tag{9}$$

where, $\widetilde{\mathbf{M}}_i$ and $\widetilde{\mathbf{K}}_i$ are the elemental mass matrix and stiffness matrix of ith element with $\rho_i = 1$, respectively; p is the penalty parameter and it is set to be p=3 in this paper. The Rayleigh damping theory is employed, and the elemental damping matrix is obtained by

$$\mathbf{C}_i = \left[\frac{\rho_{\min} - \rho_{\min}^p}{1 - \rho_{\min}^p} \left(1 - \rho_i^p\right) + \rho_i^p \right] \left(\alpha \widetilde{\mathbf{M}}_i + \beta \widetilde{\mathbf{K}}_i\right) \tag{9}$$

where, α and β are the damping parameters.

5. Sensitivity analysis scheme

The topology optimization problems always are solved by gradient-based mathematical programming algorithms, which need the sensitivity analysis of the objective function with respect to design variables. In this paper, a sensitivity analysis scheme derived by adjoint variable method is applied, which is more efficient than direct variable method in the problems involving a large number of design parameters. For the case, initial condition independent of design parameters, the sensitivity analysis scheme can be expressed as

$$\frac{\partial J}{\partial \rho_i} = \mathbf{X}(0)^{\mathrm{T}} \frac{\partial \mathbf{P}}{\partial \rho_i} \mathbf{X}(0) = \sum_{k=1}^{2N} \sum_{l=1}^{2N} \lambda_{kl} D_{kl}^i \tag{10}$$

where, $\boldsymbol{\lambda}$ is the adjoint matrix. $\boldsymbol{\lambda}$ and $\mathbf{D}$ can be obtained by

$$\mathbf{A}\boldsymbol{\lambda} + \boldsymbol{\lambda}\mathbf{A}^{\mathrm{T}} + \mathbf{S} = 0 \tag{11}$$

where, $\mathbf{S} = \mathbf{X}(0)\mathbf{X}(0)^{\mathrm{T}}$

$$\mathbf{D}^i = \frac{\partial \mathbf{Q}}{\partial \rho_i} + \frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial \rho_i} \mathbf{P} + \mathbf{P} \frac{\partial \mathbf{A}}{\partial \rho_i} \tag{12}$$

where, $\frac{\partial \mathbf{A}}{\partial \rho_i} = \begin{bmatrix} 0 & 0 \\ -\frac{\partial (\mathbf{M}^{-1})}{\partial \rho_i} \mathbf{K} - \mathbf{M}^{-1} \frac{\partial \mathbf{K}}{\partial \rho_i} & -\frac{\partial (\mathbf{M}^{-1})}{\partial \rho_i} \mathbf{C} - \mathbf{M}^{-1} \frac{\partial \mathbf{C}}{\partial \rho_i} \end{bmatrix}$

6. Transformation of equations to reduced space

When the analysis model has a large numbers of DOFs, the solution of Lyapunov matrix equation is computational costly, which will makes the computing time of optimization process increased significantly. For example, for a 1,000-dof system, the number of unknowns in P is 2,001,000. Thus model reduction is necessary to implement the proposed approach. The mode reduction method and eigenmode selection method are used to decrease the computing time of optimization process.

6.1 mode reduction method

To use mode reduction method, a linear transformation is employed, which can be expressed as

$$\mathbf{u} = \mathbf{T}\mathbf{u}_m \tag{13}$$

where, $\mathbf{u}$ and $\mathbf{u}_m$ are the displacement vectors of full model and reduced model, respectively; $\mathbf{T}$ is the transformation matrix. Generally, matrix $\mathbf{T}$ contains the first several eigenmodes of full model. However, for some cases, the eigenmodes of lower order may have no effect on the structural response. A selection method is applied to find these eigenmodes to decrease the number of basis vectors in transformation matrix to further reduce the scale of reduced model, and will be introduced in next section. The transformation matrix **T** is expressed as

$$\mathbf{T} = \left\{ \boldsymbol{\varphi}_{c_1}, \boldsymbol{\varphi}_{c_2}, \ldots, \boldsymbol{\varphi}_{c_m} \right\} \tag{14}$$

where, c_1, c_2, c_m are the number of 1st, 2nd, mth reserved eigenmodes. The mass, damping, and stiffness matrices of reduced model are respectively obtained by

$$\mathbf{M}_{re} = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T},\ \mathbf{C}_{re} = \mathbf{T}^{\mathrm{T}}\mathbf{C}\mathbf{T},\ \mathbf{K}_{re} = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T} \tag{15}$$

The initial conditions of reduced model are obtained by

$$\mathbf{u}_{re,0} = \mathbf{M}_{re}^{-1}\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{u}_0,\ \mathbf{v}_{re,0} = \mathbf{M}_{re}^{-1}\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{v}_0 \tag{16}$$

Include the sensitivity of matrix **T** with respect to design parameters in sensitivity analysis scheme will make the analysis much complicated. Thus, in this paper, the sensitivity of matrix **T** with respect to design parameters is ignored.

6.2 Eigenmode selection method

We use the model participation factor (MPF) to evaluate which eigenmode in first several eigenmodes of full model have no effect on the structural response. High value of MPF of ith eigenmode means that this eigenmode has large effect on structural response. Low MPF value means that this eigenmode has a little effect on structural response. The MPF value is obtained by

$$\mathrm{MPF} = \frac{\left(\mathbf{u}_0^{\mathrm{T}}\boldsymbol{\varphi}_i\right)^2}{\left(\mathbf{u}_0^{\mathrm{T}}\mathbf{u}_0\right)\left(\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i\right)},\ \mathrm{MPF} = \frac{\left(\mathbf{v}_0^{\mathrm{T}}\boldsymbol{\varphi}_i\right)^2}{\left(\mathbf{v}_0^{\mathrm{T}}\mathbf{v}_0\right)\left(\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i\right)} \tag{17}$$

where, $\mathbf{u}_0$ and $\mathbf{v}_0$ are the initial displacement and velocity vector, respectively, and $\boldsymbol{\varphi}_i$ is the eigenvector of ith eigenmode. The MPF values of all eigenmodes are located between 0 and 1.

For the case an optimization problem has both initial velocity and displacement, the MPF values for initial velocity and displacement need to be calculated separately and weighted summed. The weighted coefficients are the objective function values from using the initial velocity and displacement as initial condition separately.

7. Numerical example

To avoid the checkerboard phenomenon, the sensitivity filter method is used, the filter radius is 1.5. For some cases, drastic change of the design may cause that the Lyapunov equation cannot be solved. Thus, the move limit of design parameter is set to be 0.02.

In this section, a numerical example is presented to verify the sensitivity analysis scheme and the proposed approach.

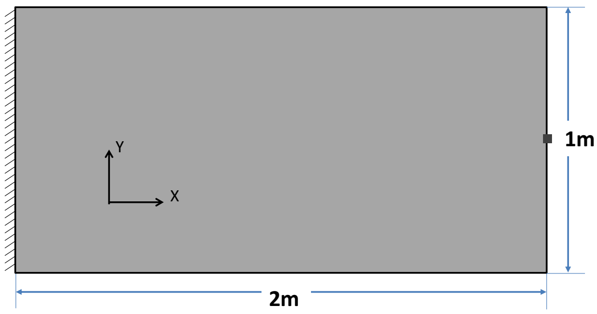


Figure 1: Geometry model

We consider a 2m×1m×0.1m rectangular plate. The left edge of the plate is clamped and other three edges are free as shown in figure 1. The material parameters is E=69GPa, ν=0.3, $\rho_i = 2700$ kg/m^3. A concentrate mass element locates at the middle of the right edge of the plate, and m=50kg. The initial condition is that the Z-direction velocity of mass element is 10m/s. The plate is uniform meshed by 4-nodes square element, 40×20, as shown in Figure 2. The objective function is

$$J = \int_0^\infty u_{mass}^2 dt \tag{18}$$

where, u_{mass} is the Z direction displacement of mass element.

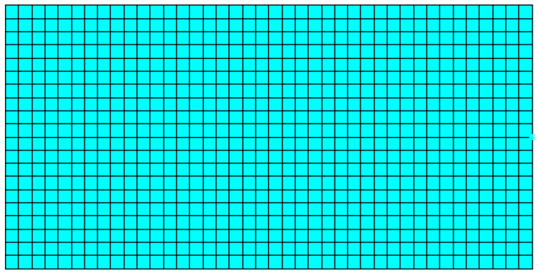


Figure 2: The finite model

Considering the symmetry of the finite element model, initial condition, and constraints, only the artificial densities of the elements in bottom half of the structure are considered in the optimization process. The artificial density of the element in the top half of structure is set to be same with that of the element at symmetrical position. Considering the accuracy of sensitivity results and efficiency of optimization process, in this example, the transformation matrix **T** will contains 60 eigenmodes selected from first 300 eigenmodes of full model.
To verify the accuracy of sensitivity results obtained by the proposed sensitivity analysis scheme, the finite difference method is also applied to obtain the sensitivity results. The sensitivity results of the purple element as shown in figure 3 by the finite difference method and adjoint method are both shown in figure 3. The damping parameters are $\alpha = 0.1$, $\beta = 0$, and the analysis model is a uniform design ($V_{frac} = 0.5$). Numerical results show that the relative error of the results obtained by two methods is small for most elements.

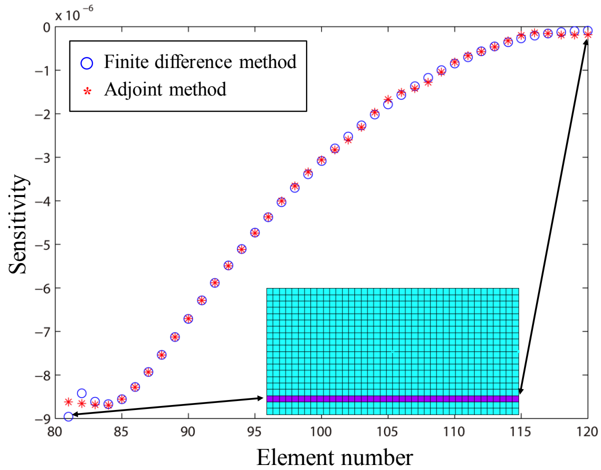


Figure 3: Sensitivity results of several elements from two methods

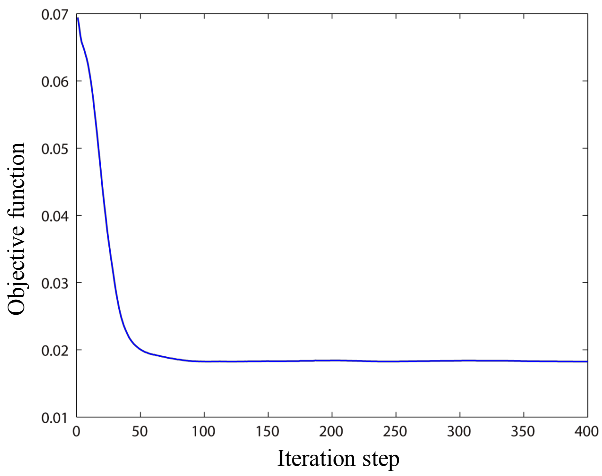
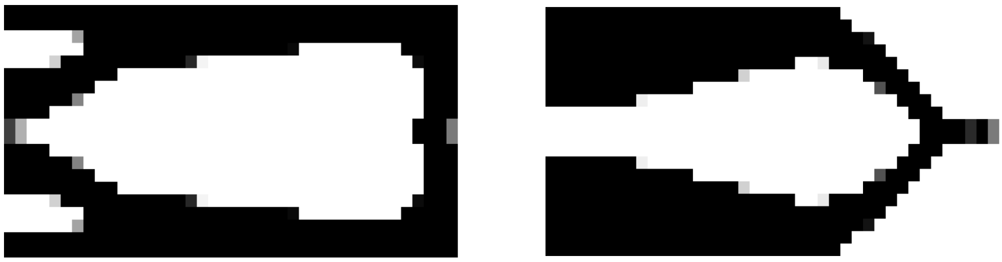


Figure 4: Iteration history of objective function

Firstly, perform a topology optimization with $\alpha = 0.1$, $\beta = 0$, $V_{frac} = 0.5$, and $\rho_{\min} = 0.001$ by proposed approach. Figure 4 shows the iteration history of objective function and. From the results, a stable decrease of the objective function can be observed. Next, perform another topology optimization with $\alpha = 0$, $\beta = 0.1$, $V_{frac} = 0.5$, and $\rho_{\min} = 0.001$ by proposed approach. The optimized designs are shown in figure 5. The results witness that the optimized designs under different damping parameters are such different. Thus, obtain the accurate damping parameters are important to whether the optimized design is reasonable.



(a) $\alpha = 0.1$, $\beta = 0$ (b) $\alpha = 0$, $\beta = 0.1$

Figure 5: Optimized designs under different damping parameters

8. Conclusions

The problem of topology optimization with respect to vibration control of a shell structure subject to initial excitation is considered. The design objective is minimization of dynamic performance index in the form of time integral of the quadratic function of state variables. An approach is developed to handle this topology optimization problem. Mode reduction method and an eigenmode selection method are applied to decrease the scale of reduced model. The numerical example is presented to verify the sensitivity analysis scheme and the proposed approach for topology optimization problem considered in this paper. The results show that the sensitivity analysis scheme for reduced model can obtain accurate results, and also witness that the damping parameters have a great effect on the optimized design.

9. Acknowledgement

This work is supported by National Natural Science Foundation of China (91216201 and 11372062).

10. References

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