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# Improvement researches on involute tooth profile

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**1. Abstract**
A kind of profile modification cubic curve of the involute spur gear is proposed in this paper using the geometric theory and the Curve fitting method. The derivation process of the key point coordinates and the curve equation is described in detail, the proposed modification curve tangents to both the involutes and the addendum circle. Using the modification curve to correct the flank shape of the driving and driven gears, the smooth transition between the cubic curve and involute can be ensured, as well as between the cubic curve and the addendum circle. Besides, the contact stress reduces obviously after modification using this method which can be verified with Hertz contact theory, finally, an example is adopted to illustrate how to implement this modification method. The goal of this article is to put forward a scheme for the optimization design and improvement of gears so as to improve the gears' working condition and prolong their service life.
**2. Keywords:** spur gears; profile modification; cubic curve; smooth transition

**3. Introduction**

It is common to see that the meshing impact, load mutation, speed fluctuation and other vibration with different formation because of the teeth deformation, manufacturing and installation errors and so on when the spur gear works[1, 2], consequently, modification was carried out on the gears in order to improve all sorts of undesirable phenomenon foregoing.

Tooth profile modification, refers to modifying the involute appropriately near the tip or root of the meshing teeth, which aims at compensating the machining error and elastic deformation of the teeth, and reducing the load impact during the gears enter and exit the mesh[3-5]. The pressure angle of the addendum of gear after modification is larger than unmodified; this means that the meshing angle of the start meshing point increases, that is the comprehensive radius of curvature of the start meshing point increases.

Height of modification, amount of modification and curve of modification are regarded as the three elements of gear modification. Amount of modification varies gradually from the maximum to zero, and its change rule is defined as the curve of modification, there are two types of modification curves: straight and curved[6], while neither straight line nor curve modification can guarantee that the modification curve not only tangent to the invonlute but also tangent to the addendum circle, in this case, when the gears enter the mesh, the addendum of driven gear contact with the root of driving gear tooth in terms of obtuse angle or chamfer, then sliding on the surface of the driving gear tooth, when the gears exit the mesh, the addendum of driving gear contact with the root of driven gear tooth in terms of obtuse angle or chamfer, then sliding on the surface of the driven gear tooth, In those cases mentioned above, the contact stress and contact deformation of gear teeth are very large, the transmission stability of gear is poor, so, the curve of modification proposed in this aims at improving all kinds of disadvantages stated above.

4.8. Tables and Figures
Tables and figures should be consecutively numbered. Place table caption above the table and figure caption below the figure. Table and figure captions should be centred. Allow one line space between the table and its caption and between the figure and its caption. Allow one line of space between the table or figure and the adjacent text.

**4.The method of calculating the modification curve**

The following part of this paper will detail the method of calculating the modification curve, In figure1, line segment $AE$ is the maximum amount of modification: $AE = \Delta_{\max}$ , involute section $AB$ is the length of the modification: $AB = L$ , the gear tooth and the addendum circle intersect at point $A$ . $h_{\max}$ is the height of modification $(mm)$ ；

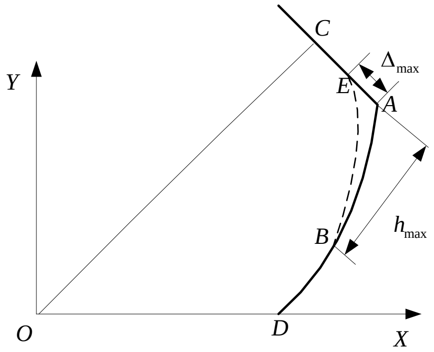


Fig.1.The three elements of gear modification

The modification curve put forward in this paper and addendum circle intersect at point E, and involute tooth surfaces intersect at point B , the modification curve tangent to the involute as well as the addendum circle at the both two points.

The specific steps of the modification curve are as follows:

4.1. Solution of point $A$ coordinate

The parameters equation of the involute can be expressed as :

$$\begin{cases} x = r_b(\cos\alpha + \alpha\sin\alpha) \\ y = r_b(\sin\alpha - \alpha\cos\alpha) \end{cases} \quad (1)$$

where $r_b$ is the radius of base circle（mm）, The radius of addendum circle can be expressed as：

$$r_a = \frac{m(z+2)}{2} \quad (2)$$

where $m$ is module（mm）, $z$ is the number of teeth, while the equation of addendum circle is

$$x^2 + y^2 = r_a^2 \quad (3)$$

from the involute equation and the equation of addendum circle, we can get that

$$r_b^2(1+\alpha^2) = r_a^2 \quad (4)$$

consequently,

$$\alpha_A = \sqrt{(\frac{r_a}{r_b})^2 - 1} \quad (5)$$

the coordinate of point $A$ is [ $r_b(\cos\alpha_A + \alpha_A\sin\alpha_A)$ , $r_b(\sin\alpha_A - \alpha_A\cos\alpha_A)$ ].

4.2. Solution of point $B$ coordinate and the slope of involute

The length of arc $BD$ is (from figure1we know that $L_{AB} \approx \frac{h_{\max}}{\cos\theta}$ )

$$L_{BD} = L_{DA} - L_{BA} = \int_0^{\alpha_A} \sqrt{(\frac{dx}{d\alpha})^2 + (\frac{dy}{d\alpha})^2}\, d\alpha - L_{BA} = \frac{r_a^2 - r_b^2}{2r_b} - \frac{h_{\max}}{\cos\theta} \quad (6)$$

while the length of arc $BD$ is ：

$$L_{BD} = \int_0^{\alpha_B} \sqrt{(\frac{dx}{d\alpha})^2 + (\frac{dy}{d\alpha})^2}\, d\alpha = \frac{r_b}{2}\alpha_B^2 \quad (7)$$

from equation（6）and（7）, the $\alpha_B$ can be obtained. The coordinate of point $B$ is [ $r_b(\cos\alpha_B + \alpha_B\sin\alpha_B)$ , $r_b(\sin\alpha_B - \alpha_B\cos\alpha_B)$ ], that is , $(x_B, y_B)$ , the slope of involute of point $B$ is

$$k_B = \frac{dy}{dx} = \tan\alpha_B \quad (8)$$

4.3. Solution of point $E$ coordinate and the slope of addendum

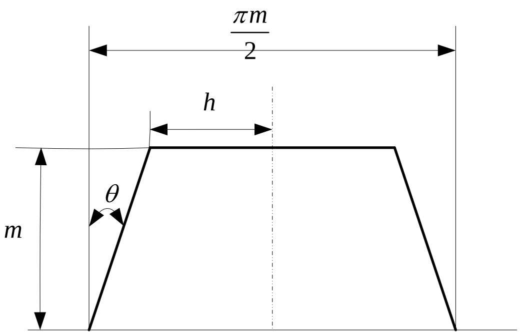


Fig.2.Gear parameter schematic diagram

As presented in figure2, the tooth surface of the addendum circle can be seen as a plane approximately, then AC is straight line, （$AC$ perpendicular to $OC$ and $C$ is the pedal）,we can easily get：

$$h = \frac{P}{4} - m\tan\theta = m(\frac{\pi}{4} - \tan\theta) \qquad (9)$$

$\theta = 20^{\circ}$ is pressure angle. In figure2, assuming $\angle AOC = \beta$ ,then $\sin\beta = \frac{AC}{OA} = \frac{h}{r_a}$ ,

$$\beta = \arcsin\frac{h}{r_a} = \arcsin\frac{m(\frac{\pi}{4} - \tan\theta)}{r_a} \qquad (10)$$

the slope of line segments $OC$ is $k_{OC} = \tan(\beta + \alpha_A)$ , and the slope of $AC$ is

$$k_{AC} = k_E = \frac{-1}{k_{OC}} = \frac{-1}{\tan(\beta + \alpha_A)} \qquad (11)$$

assuming the coordinate of point $E$ is $(x_{E,} y_E)$ ,then $k_{AC} = k_E = \frac{y_E - y_A}{x_E - x_A}$ , meanwhile, $|EA| = \Delta_{\max}$ ,that is $\sqrt{(x_E - x_A)^2 + (y_E - y_A)^2} = \Delta_{\max}$ , from above all we can get: $(x_E - x_A)^2 + k_E^2(y_E - y_A)^2 = \Delta_{\max}^2$ ,that is $(1 + k_E^2)(x_E - x_A)^2 = \Delta_{\max}^2$ , we can get fatherly：

$$\begin{cases} x_E = x_A - \frac{\Delta_{\max}}{\sqrt{1 + k_E^2}} \\ y_E = y_A - \frac{k_{AC}\Delta_{\max}}{\sqrt{1 + k_E^2}} \end{cases} \qquad (12)$$

4.4. Comprehensive above solution ,using the polynomial interpolation method, The cubic curve can be calculated that meet through points A and B ,as well as tangent to the no modification curve at the both points is as follows:

$$y = ax^3 + bx^2 + cx + d, x \in [x_B, x_E] \qquad (13)$$

various parameters are calculated as follows (the coordinate system of xOy is the coordinate system of involute):

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x_B^3 & x_B^2 & x_B & 1 \\ x_E^3 & x_E^2 & x_E & 1 \\ 3x_B^2 & 2x_B & 1 & 0 \\ 3x_E^2 & 2x_E & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} y_B \\ y_E \\ k_B \\ k_E \end{bmatrix} \qquad (14)$$

Where $x_B$、 $y_B$、 $x_E$、 $y_E$、 $k_B$、 $k_E$ are solved according to the steps mentioned above.

**5.Theoretical bases**

In this part of the paper, the gear contact stress decreases after modification is certified with Hertz contact theory [7].

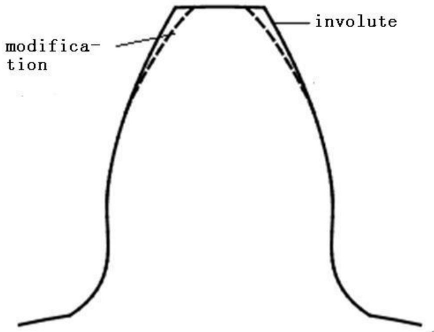

Fig.3.The modification curve diagram

In figure 3, the solid line represents the involute gear tooth profile, the dashed lines represent the tooth profile after modification using the proposed curve put forward in this paper. Based on Hertz contact theory, gear contact is approximately to regard as two cylindrical surfaces contact which have the parallel axis showed in figure 3,the driving tooth profile represented by the solid line contact with the driven gear surface in terms of obtuse angle ,the obtuse angle can be regarded as a circular arc which radius is small enough, its radius is $R_1$ ,the radius of the driven gear is $R$ , according to the Hertz contact theory:

$$\begin{cases} b_1 = \sqrt{\dfrac{4F}{\pi L}\left(\dfrac{1-\nu_1^2}{E_1}+\dfrac{1-\nu_2^2}{E_2}\right)\dfrac{R_1 R}{R_1+R}} \\ \sigma_{1\max} = \sqrt{\dfrac{F}{\pi L}\dfrac{1}{\dfrac{1-\nu_1^2}{E_1}+\dfrac{1-\nu_2^2}{E_2}}\dfrac{R_1+R}{R_1 R}} \\ \sigma_1 = \sqrt{\dfrac{F}{\pi L}\dfrac{1}{\dfrac{1-\nu_1^2}{E_1}+\dfrac{1-\nu_2^2}{E_2}}\dfrac{R_1+R}{R_1 R}}\sqrt{b^2-x^2}, (-b \le x \le b) \end{cases} \tag{15}$$

where $b_1$ is half of the contact width, $\sigma_{1\max}$ is the maximum contact stress, $\sigma_1$ is contact stress distribution along the direction of contact surface width. Similarly, the addendum of driving tooth after modification represent by dashed lines contact with the surface of driven gear in terms of circle arc with the radius of $R_2$ ,the parameters are calculated as follows:

$$\begin{cases} b_2 = \sqrt{\dfrac{4F}{\pi L}\left(\dfrac{1-\nu_1^2}{E_1}+\dfrac{1-\nu_2^2}{E_2}\right)\dfrac{R_2 R}{R_2+R}} \\ \sigma_{2\max} = \sqrt{\dfrac{F}{\pi L}\dfrac{1}{\dfrac{1-\nu_1^2}{E_1}+\dfrac{1-\nu_2^2}{E_2}}\dfrac{R_2+R}{R_2 R}} \\ \sigma_2 = \sqrt{\dfrac{F}{\pi L}\dfrac{1}{\dfrac{1-\nu_1^2}{E_1}+\dfrac{1-\nu_2^2}{E_2}}\dfrac{R_2+R}{R_2 R}}\sqrt{b^2-x^2} (-b \le x \le b) \end{cases} \tag{16}$$

where $b_2$ is half of the contact width, $\sigma_{2\max}$ is the maximum contact stress, $\sigma_2$ is contact stress distribution along the direction of contact surface width. $F$ is the normal contact force （$N$）， $L$ is the length of contact line, it is gear width here（$mm$）, $\nu_1$ , $\nu_2$ are the poison's ratio of active and passive gear materials respectively, $E_1$ , $E_2$ are the elasticity modulus of active and passive gear materials respectively（$MP_a$）,Due to the tooth surface tangent to the addendum circle after modification, thus, $R_1 < R_2$ , consequently $b_1 < b_2$ , $\sigma_{1\max} > \sigma_{2\max}$ , $\sigma_1 > \sigma_2$ , so the driving gear teeth after modification according to this cubic curve have larger contact area and less contact stress during engaging-in, while the driven gear teeth after modification according to this cubic curve has larger contact area and less contact stress during engaging-out.

Example

To describe the steps of calculating this modification curve more clearly, the following part is a specified example using the method, A spur gear has the main parameters as shown in table 1.

Table 1：Gear parameters

| | |
|---|---|
| *Number of teeth* | 43 |
| *Module*/mm | 3 |
| *Pressure angle*/deg | 20 |
| *Face width*/mm | 82 |
| *The addendum coefficient* | 1 |
| *Gear tip clearance coefficient* | 0.25 |
| *The maximum amount of modification*/mm | 0.05 |
| *The modification height*/mm | 1.5 |

According to the mentioned method, the coordinate of points $A$ , $B$ and $E$ are $(67.4553, 2.32)$ , $(65.28, 1.31)$ and $(67.4, 2.32)$ respectively. The involute slope of point $B$ is $k_B = \tan\alpha_B = 0.4228$ .the addendum slope of point $E$ is $k_E = -1.8$

According to $y = ax^3 + bx^2 + cx + d, \{x \in [x_B, x_E]\}$ , $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x_B^3 & x_B^2 & x_B & 1 \\ x_E^3 & x_E^2 & x_E & 1 \\ 3x_B^2 & 2x_B & 1 & 0 \\ 3x_E^2 & 2x_E & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} y_B \\ y_E \\ k_B \\ k_E \end{bmatrix}$ ,

The coordinates of points $B$ and $E$ as well as the slope of the modification of these two points are substitute into the equation. The coefficients of cubic curve can be solved and the cubic curve of modification we obtain is ：

$y = -0.518367x^3 + 102.6712x^2 - 6773.383x + 148968.649$ ； $\{x \in [65.28, 67.4]\}$

**6.Conclusions**

A modification curve of cubic curve is put forward in this paper, compared with the existing modification curve, the smooth transition between modification curve and involute has improved the stability of the transmission and reduce the vibration and noise, this is helpful for the improvement of the gear working conditions and increasing the service life. The smooth transition between modification curve and addendum has made the driven（driving）gear addendum contact the driving（driven）gear surface with circle arc instead of sharp corner during engaging-in（engaging-out, in this case according to Hertz contact theory, the driving and driven gear can be approximately regarded as the two cylinder with parallel axis, this increases the instantaneous contact area of the gear tooth surface, and decreases the contact stress of the gear teeth during both engaging-in and engaging-out, consequently, thus the instantaneous impact is reduced between teeth, besides, the cubic curve of modification is simple and suitable for parametric design and programming of numerical control machining.

**7. Acknowledgements**

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