

## Method of Variable Transformation for Topology Optimization with Clear Boundary Shape

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### 1. Abstract

The problem of topology optimization based on the density-based approach using gradient optimization methods is considered in the paper. The filtering procedure is used to avoid local minima and to control the topology design. Commonly this procedure is applied to the sensitivity field, since filtering of the design variables leads to strong blur solutions. To overcome these blurring an additional procedure is required. In this paper both sensitivity filter and density filter are used. But for the density filter the following new approach is applied. Transformation of variables with values from 0 to 1 to new design variables with values from  $-\infty$  to  $+\infty$  is performed. Then simple Gauss filtering is applied and reverse transformation to the original variables is fulfilled. The transformation function has the following features: for the “grey values” it is nearly linear, and at  $-\infty$  and  $+\infty$  it approaches asymptotically to the value 0 and 1, respectively. Variation statement of the problem of finding this transformation function is proposed. Also, the change of the properties of transformation function allows controlling topology layout. This approach is demonstrated on the problems of topology optimization for minimization of structural compliance at a given volume. The advantages of proposed approach and the obtained solutions are discussed.

**2. Keywords:** Topology optimization, Conservative filtering, Design variable transformation, Clear boundaries

### 3. Introduction

Topology optimization is a modern tool of structural design. In this method, the distribution of the structural material is described by design variables taking the discrete values 0 and 1. For the efficient use of gradient optimization methods it is required to switch from integer to real design variables. The general approach is the addition of intermediate values between 0 and 1. However, this leads to “gray” solutions that are difficult to interpret by designer, and besides, they may significantly differ from the optimal “clear” solutions. In common methods of topology optimization with penalization [1] the proportion of “gray” is regulated by penalization parameter  $p$ . The increase in parameter  $p$  enhances the sharpness of the design shape boundaries, but it can also increase the risk of sub-optimal design, so this parameter should be limited. One of the ways to avoid the risk of stopping the algorithm in a local minimum is to use filtering procedure. On the other hand, filtering also increases the proportion of “gray”. It is necessary to find a compromise between the increase in penalization parameter and the degree of filtering. There are some methods of filtering, which reduce the side effect of blurring. Firstly, it is due to the filtering of the derivative of the objective function with respect to the design variables (sensitivity) instead of filtering of design variable values. Secondly, the filtering result is subjected to further processing, for example, by application of projection method [2]. This projection filtering method requires additional computations and significantly increases computational costs. There is also the problem to satisfy the specified constraints. To overcome this difficulty related procedure has become quite complicated [3]. An alternative approach is shown in [4]. This approach is accomplished by a transition to an infinitely large penalization parameter and by an introduction of new design variables which belong to the range  $(-\infty, \infty)$ . Filtering procedure in this method is carried out on these new variables. This method allows getting closer to the optimal design with clear boundary shape at low computational cost. The problem of finding the best transformation function to the new design variables is considered in the paper.

### 4. Conservative filtering

We introduce a new design variables  $z$  of the computational domain  $(-\infty, \infty)$ , which are associated with the design variables  $y$  of the physical domain  $[0, 1]$  as follows:

$$y = \frac{1}{2}(1 + f(2z)), \quad (1)$$

$$z = \frac{1}{2}f^{-1}(2y - 1) \quad (2)$$

In paper [4] function  $f(z)=\tanh(z)$  is used. Not formally speaking, this transformation does not change the “gray”

values because the mapping is close to the identity, while the “white” and “black” values vary significantly. Filtration of the  $y$  values is performed in three stages. First, turn to the computational domain, according to Eq.(2), then carry out Gaussian filtering of distribution of values  $z$ , and return to the physical domain, according to Eq.(1). Let us demonstrate the result of filtering in the one-dimensional and two-dimensional cases. Figure 1 shows the result of one-dimensional filtering case with the initial stepwise distribution:

$$y(t) = \begin{cases} \varepsilon_{\min} & , t < 0 \\ 1 - \varepsilon_{\max} & , t \geq 0 \end{cases}$$

With the same Gaussian filter radius  $r_s$ , three sets of values  $\varepsilon_{\min}$  and  $\varepsilon_{\max}$  is considered: 1)  $\varepsilon_{\min}=\varepsilon_{\max}=\varepsilon_2$ , 2)  $\varepsilon_{\min}=\varepsilon_2$ ,  $\varepsilon_{\max}=\varepsilon_1$ , 3)  $\varepsilon_{\min}=\varepsilon_{\max}=\varepsilon_1$ . Values  $\varepsilon_1$  and  $\varepsilon_2$  satisfy the inequalities  $0 < \varepsilon_1 < \varepsilon_2 < 1$ . These examples demonstrate that the closer to the boundary values, the less blur. We also call attention to the possibility of shifting the position of step without significant blurring (Figure 1b).

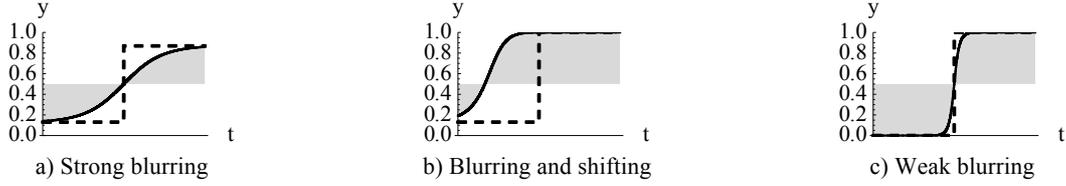


Figure 1: Filtering of stepwise distribution in new design variables

Figure 2 shows the result of filtering in two dimensions with different radii  $r_s$ . The initial distribution contains three sets of values  $\varepsilon$ ,  $0.5$ ,  $1-\varepsilon$ ,  $\varepsilon \ll 1$  submitted by the shades of gray. The boundaries between these values vary significantly in various ways. The boundary between the black and gray, as well as the boundary between gray and white is not preserved, it is blurred and shifted so that the gray area is significantly reduced when large radii filtration. On the contrary, the boundary between black and white is still the same, it is not blurred and does not move. The dashed line is applied for clarity, to emphasize this effect.

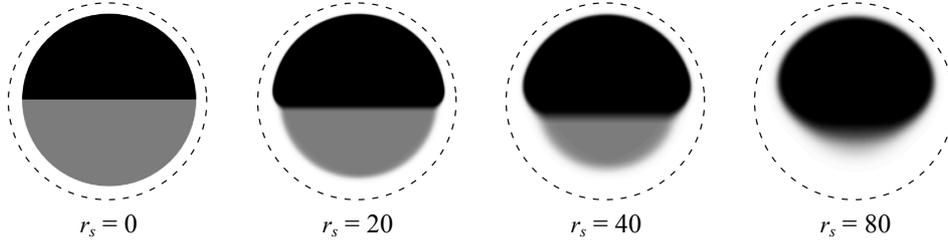


Figure 2: Conservative filtering with different radii  $r_s$

Since this filter keeps formed boundaries of the structure, it can be called conservative filtering.

## 5. The choice of transformation function

The transformation function  $f$ , used in the Eq.(1) and Eq.(2), must have the following properties. For simplicity, we consider only the antisymmetric functions and therefore we consider the range from  $0$  to  $\infty$ , and for negative values of  $z$  we redefine the function as follows  $f(z) = -f(-z)$ . The function must be monotonically increasing, have a linear region near zero, asymptotically fast enough approaches the limit values  $-1$  and  $1$ , it is necessary for conservative filtering effect. From heuristic reasons, we assume that the best function is the least different from linear in all computational domain. A measure of the difference can be written as functional. These requirements lead us to the variation formulation of finding the optimum transformation function

$$\begin{cases} F(f, m, \alpha) = \int_0^{\infty} \frac{f^{(m)}(x)^2}{f'(x)^\alpha}, & \alpha \geq 0 \\ f'(x) \geq 0, f'(0) = 1, f(\infty) = 1, f^{(2k)}(0) = 0, k = \overline{0, [(m-1)/2]} \end{cases} \quad (3)$$

Meaning of the parameter  $\alpha$  is that it allows us to control the speed of the asymptotic approaching, and at the same time the value of  $\alpha > 0$  guarantees a monotonic increase function. Two cases  $m=2$  and  $m=3$  for  $\alpha=1$  are considered. In the case  $m=2$  the functional Eq.(3) attains its minimum on the extremal

$$f(z) = 1 - e^{-z} \quad (4)$$

This function has a non-zero second derivative; therefore, the redefined function has a discontinuous second derivative at zero and is not smooth enough. In examples discussed in [4], it was found that the results of the function Eq.(4) are close to  $f(z)=\tanh(z)$ .

Smoother solutions are obtained in the case  $m=3$ , the functional Eq.(3) attains its minimum on the extremal

$$f(z) = \frac{7 + 4\sqrt{3} - e^{-\sqrt{3}z}}{(2 + \sqrt{3} + e^{-\sqrt{3}z})^2} (1 - e^{-\sqrt{3}z}) \quad (5)$$

For computational efficiency it is required to calculate quickly the derivative of the function  $f$ , but for Eq.(5) the derivative is rather cumbersome. It turned out that there is a good approximation with a simple derivative, yielding on some test cases indistinguishable results. Five variants of transformation functions are presented in Table 1 with their asymptotic rate and values of the functional. The value of the functional on the extremal is a measure of the accuracy of approximation for other functions. The function  $f_3$  is the best approximation function for both extremal  $f_4$  and  $f_5$  with last it is almost identical:  $F(f_3,2,1) - F(f_5,2,1) = 0.2337$ ,  $\max(f_3(z)-f_4(z)) = 0.1094$   $F(f_3,3,1) - F(f_5,3,1) = 0.04403$ ,  $\max(f_3(z)-f_5(z)) = 0.005752$ .

Table 1: Transformation functions and their properties

$i$	1	2	3	4	5
$f_i(z)$	$\operatorname{erf}\left(\frac{\sqrt{\pi}}{2} z\right)$	$\tanh(z)$	$\frac{2}{\pi} \operatorname{gd}\left(\frac{\pi}{2} z\right)$	$1 - e^{-z}$	$\frac{7 + 4\sqrt{3} - e^{-\sqrt{3}z}}{(2 + \sqrt{3} + e^{-\sqrt{3}z})^2} (1 - e^{-\sqrt{3}z})$
$\frac{df_i}{dz}$	$e^{-\frac{\pi}{4}z^2}$	$\operatorname{sech}(z)^2$	$\operatorname{sech}\left(\frac{\pi}{2} z\right)$	$e^{-z}$	$\frac{9 \cosh(\sqrt{3}z) + 6\sqrt{3} \sinh(\sqrt{3}z)}{(1 + 2 \cosh(\sqrt{3}z) + \sqrt{3} \sinh(\sqrt{3}z))^2}$
$\lim_{z \rightarrow \infty} \frac{1 - f_i(z)}{z}$	$O\left(z^{-1} e^{-\frac{\pi}{4}z^2}\right)$	$O(e^{-2z})$	$O\left(e^{-\frac{\pi}{2}z}\right)$	$O(e^{-z})$	$O(e^{-\sqrt{3}z})$
$F(f_i,2,1)$	1.5708	1.33333	1.2337	1	1.23594
$F(f_i,3,1)$	4.9348	3.2	3.04403	-	3

In the future, all the calculations are carried out for  $f=f_3$ , unless otherwise is specified. Then the expressions Eq.(1) and Eq.(2) take the form

$$y = \frac{2}{\pi} \arctan(e^{\pi z}),$$

$$z = \frac{1}{\pi} \ln\left(\tan\left(\frac{\pi}{2} y\right)\right)$$

## 6. The problem of topology optimization

Simple and proven method of topology optimization of structure is based on the minimum compliance problem of the structure

$$\min C(x_1, x_2, \dots, x_n) = \mathbf{f}^T \mathbf{u}$$

subjected to

$$\mathbf{K} \mathbf{u} = \mathbf{f}$$

$$V(x) = \sum_{i=1}^n x_i = V_0$$

where  $C$  – the potential strain energy (compliance),  $x_i = 0$  or  $1$  – the design variables,  $i = 1, \dots, n$ ,  $n$  – the number of finite elements,  $V_0$  – the given volume,  $\mathbf{u}$  – the vector of displacements,  $\mathbf{K}$  – the global stiffness matrix and  $\mathbf{f}$  – the vector of forces. In common method SIMP [1] an artificial power law is used (penalization) with parameter penalization  $p > 1$  of the elastic properties of the material from the design variables  $E_i(x_i) = x_i^p E_0$ , where  $E_0$  is Young modulus. In this case  $C(y_1, y_2, \dots, y_n)$  is minimized and is identical with the initial goal. However, now the constraint on the volume is nonlinear function of design variables

$$V(y) = \sum_{i=1}^n y_i^{1/p} = V_0$$

As for intermediate values  $y_i$  the inequality  $\sum_{i=1}^n y_i < V_0$  is valid, one can choose a reduced value for the volume optimization step, which will lead to the same result as for the optimization step in the initial formulation. Thus, the solution obtained with  $p > 1$ , coincides with the solution for  $p = 1$  (without penalization), but with the reduced value of volume. The idea of using a reduced value of the volume lies in the fact that the gradual removal of an important element for the structure comes a moment when the sensitivity of properties of the structure sharply increases (a derivative of compliance) with respect to the design variable corresponding to this element. The presence or absence of growth of sensitivity is the criterion that this element must be left or removed. The problem of minimum compliance with the constraint on the volume is solved by conventional way. The Lagrange function is used  $L=C-\lambda(V-V_0)$ , where  $\lambda$  is the Lagrange multiplier. Initial data is specified for design variables and then they are changed on the step  $h$

$$y^{new} = y - h \frac{\partial L}{\partial y}. \quad (6)$$

The sensitivity with respect to the element is

$$\frac{\partial C}{\partial y} = 2\mathbf{u}^T \frac{\partial \mathbf{f}}{\partial y} - \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial y} \mathbf{u}.$$

The derivative of volume constraint is

$$\frac{\partial V}{\partial y} = \frac{1}{p} y^{\frac{1}{p}-1}.$$

Thus

$$\frac{\partial L}{\partial y} = \frac{\partial C}{\partial y} - \lambda \frac{1}{p} y^{\frac{1}{p}-1}.$$

This expression can proceed to the  $p \rightarrow \infty$ . Dependence on  $y$  is simplified to

$$\frac{\partial L}{\partial y} = \frac{\partial C}{\partial y} - \frac{\lambda}{y}, \quad (7)$$

where instead of  $\lambda/p$  is written simply  $\lambda$  because its value is still required to determine. By multiplying by  $y$  the speed of design variable change is slowed when approaching to  $y=0$  so that instead of Eq.(7) the following expression is used

$$\left. \frac{\partial L}{\partial y} \right|_{delay} = y \frac{\partial C}{\partial y} - \lambda \quad (8)$$

Transfer to computational domain  $z$  is performed with agreement to Eq.(1) and Eq.(2). Control parameters *scale* and *shift* of linear transformation of argument of derivative.

$$\left. \frac{dy}{dz} \right|_{control} = \left. \frac{dy}{dz} \right|_{z \leftarrow scale \cdot z + shift}$$

Instead of Eq.(6) we obtain the following expression:

$$z^{new} = z - h \left. \frac{\partial L}{\partial y} \right|_{delay} \left. \frac{dy}{dz} \right|_{control} \quad (9)$$

Control parameters influence on speed of change for design variables on bounds  $y=0$  and  $y=1$ . Taking Eq.(8) instead of Eq.(9) we use expression, including filtering both sensitivity and design variables:

$$z^{new} = \langle z \rangle - h \left( \left\langle y \frac{\partial C}{\partial y} \right\rangle - \lambda \right) \left. \frac{dy}{dz} \right|_{control} \left( \langle z \rangle \right),$$

where  $\langle \rangle$  means application of Gaussian filtering. Computational costs are sufficiently reduced, because 100

steps of renewing of design variables are performed for one step of renewing of sensitivity values. The Lagrange multiplier  $\lambda$  is determined for new values of  $z^{new}$ . Since the constrain  $V(y)=V_0$  is degenerated at  $p \rightarrow \infty$ , then the value of  $\lambda$  can be found from heuristic considerations  $z_{n_0}^s + z_{n_0+1}^s = 2z_0$ . Here  $z^s = sort(z^{new})$  is sorted list of new values for design variables  $z^{new}$ , the index  $n_0$  is equal to the number of removing elements and  $z_0 = y^{-1}(y_0)$ , where  $y_0$  is predetermined value (Figure 3).

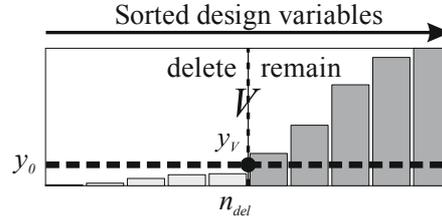


Figure 3: The criteria for determination of the Lagrange multiplier from the constraints on given

### 7. Numerical optimization results

Two-dimensional problem is considered, in which the initial structure is modeled by quadrilateral isoparametric finite elements of 2D theory of elasticity. All elements have constant thicknesses and equal 1. Initial domain is rectangle. All physical values are specified in non-dimensional manner. Poisson's ratio is 0.3, given volume  $V_0=0.5$  and nominal Young's modulus  $E_0=1$ . The design domains are given below with boundary conditions and loads together with optimization results for typical examples. In first two examples the concentrated force is applied. In the third one the distributed structural weight loads are considered. It was shown that it is possible to control the topology layout complexity by using parameters  $y_0, r_s, scale, shift$ . The results are presented in ascending order of compliance value  $C$ . Number of iterations  $Niter$  differs significantly for these cases. Note that the last structure in each example does not have holes. So, structural topology optimization is equivalent to shape optimization. The results for MBB beam for different values of control parameters are given in Table 2.

Table 2: MBB beam results

$C$	94.2125	95.1154	101.275	117.588	285.886
$Niter$	436	87	247	122	82
$y_0$	0.5	0.3	0.5	0.5	0.1
$r_s$	1.	1.	1.	1.	1.
$scale$	0.5	0.8	0.5	0.5	0.5
$shift$	0.75	0.5	0.78	0.8	1.3

The results for cantilever for different values of control parameters are given in Table 3. Note that the nonsymmetrical solutions can be obtained for symmetrical formulation (third structure). Also here it is possible to control optimization result, if the non-uniform initial distribution of design variables is specified. This is shown in the low row for third and last structure. The number of iterations significantly depends on the initial values, e.g. only 42 iterations are needed for the last structure, on contrary for the last but one it is needed 765 iterations.

Table 3: Cantilever results

$C$	180.223	183.432	186.639	187.067	195.328	201.315	207.835	254.127	635.884
$Niter$	57	84	470	232	377	289	664	765	42
$y_0$	0.3	0.3	0.0025	0.3	0.3	0.04	0.02	0.02	0.3
$r_s$	0.25	0.5	0.25	0.6	0.6	0.5	0.3	0.3	0.25
$scale$	1.	0.9	0.7	0.7	0.7	0.7	0.5	0.5	0.5
$shift$	0.03	0.5	0.7	0.98	1.05	1.4	2.	2.1	1.

The values of compliance in the considered examples are lower than the obtained ones in [4]. The results for beam

with hinged movable support (Figure 4a) and hinged immovable support (Figure 4b) under gravity forces for different control parameters.

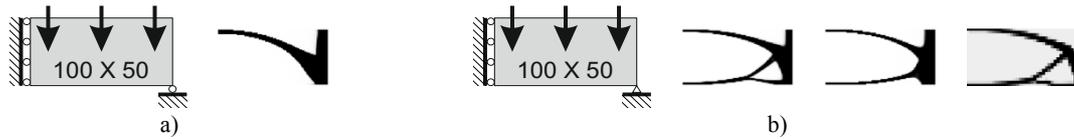


Figure 4: Beam under self-weight

This example brightly demonstrates a necessity of obtaining clear boundary shape. Here we got significantly different structure if compared with the result from paper [5]. It is last picture in Figure 4b.

## 7. Conclusion

New method of variable transformation for topology optimization for obtaining clear boundary shape of structures has been proposed. This method was demonstrated on the problems of topology optimization for minimization of structural compliance at a given volume. The advantages of proposed approach and the obtained solutions are discussed. All obtained solutions have clear boundary shape. Another advantage of the method is its computational efficiency in comparison with known SIMP methods with continuation and Heaviside projection filter.

## 6. References

- [1] M.P. Bendsoe and O. Sigmund, *Topology Optimization: Theory, Methods and Applications*, Springer-Verlag, Berlin, 2003.
- [2] J.K. Guest, J.H. Prévost, T. Belytschko, Achieving minimum length scale in topology optimization using nodal design variables and projection functions, *International Journal for Numerical Methods in Engineering*, 61(2), 238–254, 2004
- [3] J. K. Guest, A. Asadpoure, S.-H. Ha, Eliminating beta-continuation from Heaviside projection and density filter algorithms, *Structural and Multidisciplinary Optimization*, 44, 443–453, 2011
- [4] V.M. Uskov, K.A. Balunov, Method for topology optimization with clear boundary shape of structure, *International Conference on Engineering and Applied Sciences Optimization (OPT-i)*, Kos, Greece, 2014 (ISBN:978-960-99994-5-8).
- [5] M. Bruyneel and P. Duysinx, Note on topology optimization of continuum structures including self-weight, *Structural and Multidisciplinary Optimization*, 29, 245–256, 2005.