Optimal Design of a Parallel Beam System with Elastic Supports to Minimize Flexural Response to Harmonic Loading

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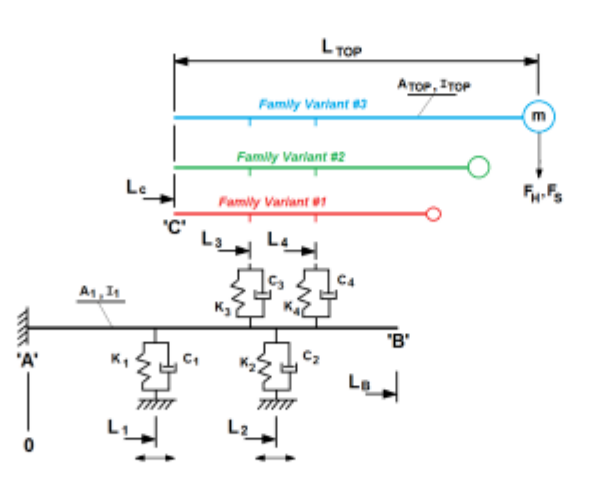
1. Abstract

Mechanical systems subject to vibration are prevalent across many industries. Although potentially different in application, they sometimes share the need to minimize aspects of flexural deformation given harmonic loading and the need to consider a variety of design variable and response-based constraints in the process. Practical design efforts also sometimes include the need for consideration of the optimal response of a platform-style product, including responses of multiple design variants supported by a common base structure. Harmonic problems can be especially challenging to optimize due to the likelihood that the response will be multi-modal; influenced by system natural frequencies throughout the design space. Further, analysis of these systems often involves large and complex computer models which require significant resources to execute. A harmonically loaded, platform-style parallel beam system with multiple family variants is used as an example in this work to demonstrate a proposed method for identifying an optimum in a constrained, multi-modal response environment with consideration for Expensive Black Box Functions (EBBF).
The presented method leverages benefits of a combined approach where the domain is first surveyed for potential areas of optimal response using a method of Steepest Feasible Descent (SFD), followed by a search in the optimal region using direct search methods. The method of SFD is a modification of the classical method of Steepest Descent, made useful for constrained models by a penalty system including both deterministic and programmatic methods. A sensitivity-based search vector method also helps to manage situations where significant difference in magnitude exists among the design variables. Evidentiary support for these key program elements is provided using standardized test functions. The effectiveness of the method is demonstrated by seeking a minimum flexural response for a parallel beam system subject to elastic support and response constraints.
2. Keywords: harmonic optimization, parallel beam, elastic supports

3. Introduction

Figure 1 illustrates the problem under study; a harmonically loaded parallel beam system with elastic supports and three (3) family variants, subject to harmonic loading through a range of frequencies. The objective of the study is to minimize flexural deformation of the tip mass ‘m’ subject to location constraints of the elastic supports as well as a maximum allowable static deformation of the tip masses ‘m’. A total of ten (10) design variables are considered.



Design Variables
L_1, K_1, C_1
L_2, K_2, C_2
K_3, C_3
K_4, C_4

Figure 1: Elastically supported parallel beam structure with 3 family variants

The family variants differ in upper-beam definition only, each constrained with a specified and differing length, tip

mass and cross-sectional geometry. As shown, the location of intermediate supports (L_3 and L_4) are fixed and represent a design constraint often encountered in platform-style products; the need for common interface design. The objective function (flexural response) is multi-objective; being comprised of the summed effect of all family variants (upper beams) upon the integrated response across the prescribed frequency range as well as the maximum range of response across the frequency range.
A common approach in the design of vibrating systems is to stiffen the structure such that the fundamental natural frequency is higher than the operating range. As is demonstrated later with this example, such a philosophy is not always possible given the design constraints and an alternate method is needed. In his text on the subject, Den Hartog [1] discusses the use of a damped dynamic vibration absorber (DDVA) as a means to reduce the magnitude of the response near to the natural frequencies. The DDVA is illustrated in Figure 2 and the Equation of Motion given in Eq.(1) below.

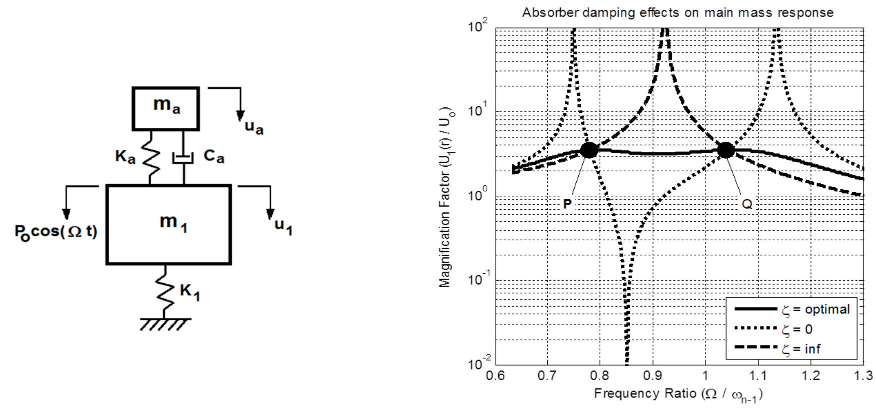


Figure 2: Damped dynamic vibration absorber

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_a \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_a \end{Bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_a \end{Bmatrix} + \begin{bmatrix} k_1 + k_a & -k_a \\ -k_a & k_a \end{bmatrix} \begin{Bmatrix} u_1 \\ u_a \end{Bmatrix} = \begin{Bmatrix} p_o \\ 0 \end{Bmatrix} \cos(\Omega t) \quad (1)$$

Also illustrated in Figure 2 is the frequency response plot for the DDVA, illustrating the effect of damping coefficient upon the system. As described by Den Hartog [1], an optimal response exists for some finite value of damping coefficient whereby the response is minimized; the slope of which becomes horizontal at frequency points ‘P’ and ‘Q’. It is this type of system behaviour that is sought for the parallel beam system in order to minimize flexural response among the family variants given that some resonant conditions may exist within the frequency range of interest. Since the parallel beam system is complex relative to the DDVA of Figure 2, a numerical optimization approach is needed to identify the optimal values.
Multiple strategies exist for optimization of such a system. Among the simplest are ‘First-Order’ methods, including the method of Steep Descent (SD) [2]. These gradient-based methods are known to be initially productive, but overall inefficient as the solution nears the optimal result. In addition, they are useful for single objective and unconstrained searches; neither of which applies to the parallel beam problem at hand. Fliege and Svaiter [3] however, propose using the method of SD for multicriteria optimization as well as adaptation of Zoutendijk’s [4] method of feasible directions for use in constrained cases. They conclude though that since the result is a first-order method, it should be considered only as a ‘first step’ toward an overall efficient method rather than an efficient method unto itself. ‘Second-Order’ methods improve upon first-order methods by incorporating Hessian matrix information and result in a more efficient process [2]. This information however is not readily available for Black Box methods [5]. Since a goal of this effort is to find a method suitable to Expensive Black Box Functions (EBBF’s), second-order methods are not considered further.
Direct methods including Genetic Algorithm (GA) [6], Particle Swarm Optimization (PSO) [5] and Sequential Quadratic Programming (SQP) [2] are advantageous in that they are suitable for constrained functions, but are known to potentially require a high number of function evaluations, particularly for multi-modal responses [2], making them undesirable for EBBF’s. In addition, although SQP, is known as a more efficient method than first-order methods, it is primarily a ‘local’ search tool with respect to multimodal response in that it has the potential to be ‘constrained’ by local maxima.[2] Laskari et al. [5], compare the use of PSO as a means of optimizing minimax problems to SQP. They conclude that for Black Box functions where gradient information is not available (as with EBBF’s) that PSO may be a good alternative as an initial search tool with continued optimization performed by more efficient methods such as SQP.

Laskari et al.'s conclusion [5], together with Fliege and Svaiter's similar conclusion regarding SD [3] is the basis for the proposed method here. That is, that the best overall method may be to initially investigate the design space using a first-order method and then, from the region of most promising minimum response, continue the search using SQP. A first order method, when limited to a few jumps and modified for use with constraints, is theorized to be more efficient for the initial search than direct methods. In addition, the use of a polynomial approximation during the steepest descent's 1-D search is theorized to be effective in identifying 'global' minima in a multi-modal environment.

4. Development and validation of proposed optimization method

The proposed optimization method uses a derivative of the first-order method of SD as the initial search method in order to make it effective for constrained searches. The derivative, termed Steepest Feasible Descent (SFD), features a deterministic penalty system as well as programmatic considerations to assure feasibility of the result from the 1-D line search. Also, consideration is given to orientation of the search vector to assure that only feasible space is searched. Finally, weighting is given to the search vector with respect to differences in order of magnitude among the design variables in order to improve effectiveness of the search.

4.1. Steepest Feasible Descent as a constrained search tool

Feasibility is considered during the execution of the 1-D line search in multiple ways. First, the length of the line search is limited by design variable constraint bounds (both side bounds and other). Secondly, as proposed by Vanderplaats [2], an external penalty term is combined with the function value to form a penalized function value to be used in the objective function. This is shown in Eq.(2) below.

$$f(x)_{pen} = f(x) + \alpha_p \sum_{j=1}^{n} \left(max\left[0, g_j(x)\right]^2 \right) \quad (2)$$

By Eq.(2), a penalty is assessed to the function value only if constraints are violated (and scaled by the multiplier $\alpha_{p)}$, which is intended to aid the 1-D line search in identifying only those minima that are in feasible space. However, as illustrated in Figure 3 below, it remains possible that a local minimum of the penalized function value could be infeasible; particularly with a multi-modal response. This is due to the effect of the squared term in Eq.(2) which minimizes the penalty near the constraint bounds.

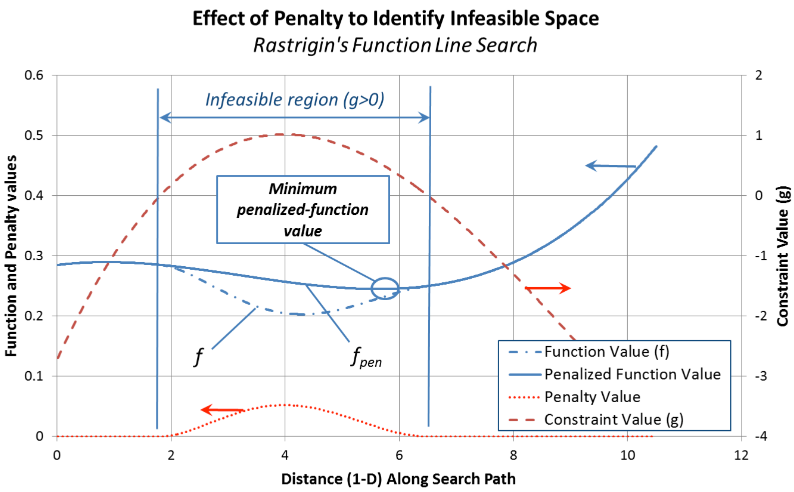


Figure 3: Identification of constraint boundaries

It is recognized that this effect could be limited by converting the penalty term in Eq.(2) to a linear function. However that would potentially cause a more abrupt transition in the resulting penalized function value at the transition from feasible to infeasible space, leading to numeric difficulties in the optimization which could be problematic. Therefore, a programmatic element is incorporated in the event that a minimum (f_{pen}) is identified within infeasible space to search the 1-D vector for a minimum penalized function value that is feasible.

In addition to considerations during the 1-D line search, feasibility is considered in the determination of the descent vector by a programmatic implementation of the Method of Feasible Directions (MFD) [2]. One common implementation of MFD is to incorporate an offset or 'push-off factor' [2] to the search vector in order to avoid a constraint bound. In the case of a nonlinear constraint bound however, an additional 'optimization exercise' is needed within the larger optimization effort [2] to avoid infeasible space. Given consideration to use of this tool for cases where multiple, highly non-linear constraints may exist; a programmatic implementation of the method of feasible directions is selected to avoid a potentially large subproblem. A minimum length for the 1-D search vector is established (as a percentage of the bounded design space). If the distance from starting point of the search vector (X0) to the nearest constraint bound (along the search vector) is measured to be below this limit, then the vector direction is programmatically modified by eliminating the coordinate term which points most directly to the nearby bound. In so doing, the resulting search vector is redirected to a trajectory 'approximately parallel' to the subject constraint bound. In this way, the new descent vector is guaranteed to provide at least a minimum search length within feasible space.

4.2. Sensitivity-based search vector

As experienced with the parallel beam problem, design variables may have significantly different orders of magnitude (length vs. spring stiffness vs. damping coefficient). If the gradient for the search vector is determined via a finite difference approach, as may be typical for Black Box functions, then an error in vector sensitivity could occur. That is, if a common step size were used for the finite difference calculation that is appropriate to the smaller variable, then it could be so small as to have an insignificant effect upon a larger variable's effect. By scaling the finite difference step size (in the direction of each given design variable) to the magnitude of the given variable, then the likelihood is increased that the descent vector will be sensitive to the impact of each variable. Eq.(3) below illustrates one method of determining such a scale effect upon the finite difference step size in a given design variable direction.

$$\Delta_x = \alpha_a \left(\frac{x_{Upper\ Bound} - x_{Lower\ Bound}}{2}\right), \quad \alpha_a = 0.5\% \tag{2}$$

4.3. Confirmation of optimization method

It is proposed that a 'combined' search methodology of SFD followed by SQP is an efficient overall search tool by leveraging the strengths of each individual method. To challenge this theory and determine if the combined method is truly better than either of the methods used individually, a test was conducted using four (4) standard test functions [7] where theoretical global optimums are known; De Jong, Rosenbrock, Rastrigin and Schwefel. For each test function, the proposed 'combined' search methodology as well as each of the component methods were run from an array of 75 starting points across the 2-dimensional design space, determined using MATLAB's 'haltonset' quasi-random method. For each case, the coordinate location of the global optimum, optimal function value and number of function evaluations was recorded. Results are shown in Table 1 below.

Table 1: Unconstrained test results

	De Jong's	Rosenbrock's	Rastrigin's	Schwefel's
Coordinate Location of Overall Minimum				
Theoretical	(0,0)	(1,1)	(0,0)	(420.969,420.969)
SFD	(4.2e-4, -4.5e-4)	(1.0067, 1.0135)	(-0.0070, 0.0117)	(421.109, 419.456)
SQP	(7.4e-6,1.0e-3)	(0.9793, 0.9597)	(4.7e-5, 1.0e-3)	(-296.88, 438.27)
Combined	(3.9e-4, -1.4e-3)	(1.0065, 1.0135)	(6.3e-4, 7.7e-4)	(421.109, 419.458)
Function Value of Overall Minimum				
Theoretical	0	0	0	0
Steep Descent	0	0	4.6e-4	2.0e-4
Direct (SQP)	2.03e-8	1.55e-7	2.7e-6	0.107
Combined	4.3e-8	1.7e-8	2.5e-6	1.9e-4
Number of Function Evaluations				
Steep Descent	1131	1190	2233	1869
Direct (SQP)	1344	4863	2118	450

Combined	1137	1194	2250	1875

With consideration for EBBF's, a minimum Design Variable step size was established as a stopping criterion, the order of magnitude of which might be considered meaningful with respect to practical design (0.001% of design variable magnitude). The error of the individual SQP solution for the Schwefel function is attributed to this fact, proof of which is not presented here due to brevity. However, a reduced value resulted in a result of similar accuracy to SFD or the Combined, but at the expense of significantly greater function evaluations.

For the multi-modal responses of Rosenbrock and Schwefel then, the proposed 'combined' search method yielded the most accurate result of either of the two (2) individual methods, while also utilizing approximately the fewest of the function evaluations of the individual methods. No practical benefit is demonstrated for De Jong's unimodal function and the proposed method is slightly less efficient (5.8%) for Rastrigin. Together, these results confirm that the combined method leverages the 'best' qualities of either of the individual methods for some multi-modal responses (with consideration for use with EBBF's), without significant consequence for the other functions.

Constrained response of the proposed method was evaluated similarly using the same test functions [7], with the addition of a circular region of infeasibility centered at the location of theoretical global optimum. Results for De Jong's and Rosenbrock's functions are shown in Figure 4 below. For each test, the location of the (same) 75 start points is identified as are the optimization path and optimum solution for each start point's search. The location of the global optimum is also identified. As shown, the modifications incorporated to the SFD approach, as well as use of the follow-on SQP method successfully prevent solutions from being identified in the infeasible region.

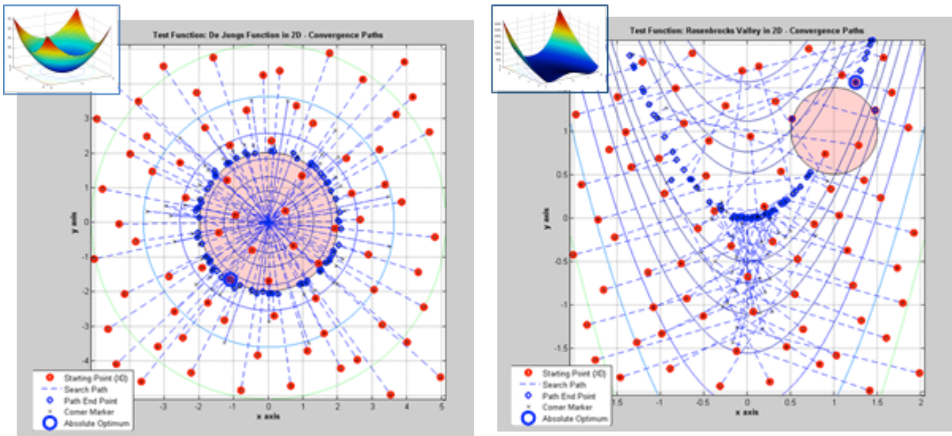


Figure 4: Constrained test results

5. Parallel beam optimization

The parallel beam problem was solved using the proposed 'combined' method with multiple start points. A 'single' objective function was defined for multi-objective use by summing combined responses of tip deflection across frequencies and for each family variant as well as the range across both frequency and model. In addition to side-bound constraints, location constraints were established for each of the supports to address the practical need that a minimum spacing must be allowed for the physical attachment structures. A response-based constraint was also considered for the static deflection of the upper beam in order to assure a minimum stiffness to the structure; to prevent the optimal harmonic design from being so flexible as to be impractical.

As indicated previously, the global SFD search was conducted from multiple start points, resulting in many areas of potential optimum throughout the design space. The best result from the among these multiple SFD results was used as the start point for the subsequent SQP search, resulting in an optimal design with relative positioning of the supports as shown in Figure 1. That is, $L_1<L_3$ and $L_3<L_2<L_4$. For purposes of comparison, frequency response plots for the 'worst' of the global SFD searches as well as the global optimal result are shown in Figure 5. Note that these data include response for each of the family variants.

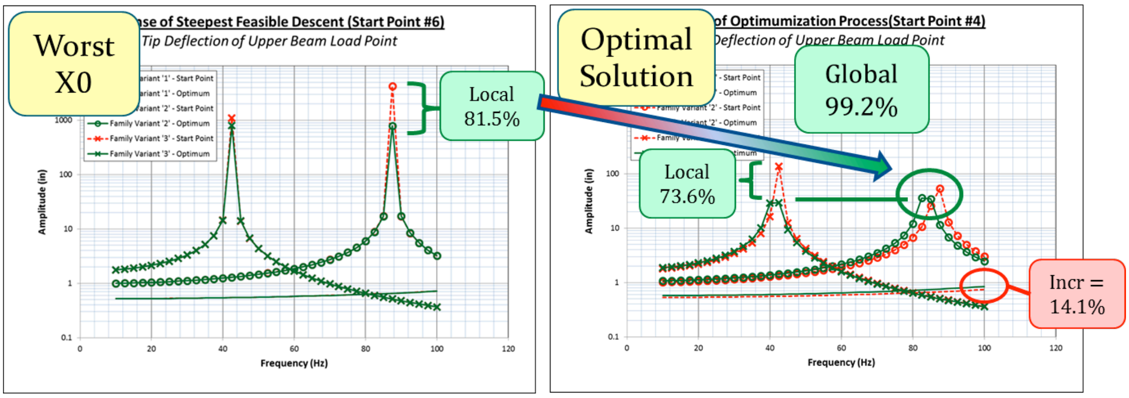


Figure 5: Parallel Beam Results

As shown, the global optimal solution results in an approximate 99.2% reduction in peak amplitude as compared to the 'worst' starting point. However, natural frequencies remain within the range of interest for two (2) of the family variants. Although not shown for brevity, these resonant responses are the fundamental frequency of the upper beam(s). Given the design constraints on upper beam specification, these frequencies could not be significantly altered. However, the resonant responses were significantly modified as predicted by Den Hartog's [1] explanation of the DDVA in Figure 2. The optimal solution is also shown to be a 'compromise' in that, although significant reductions were made in the resonant responses, the third (stiffest) family variant worsened slightly in the process. This highlights an important aspect of platform-style design, that 'compromise' solutions must be considered and managed in the optimization process.

6. Conclusions

A proposed 'combined' optimization method utilizing the method of Steepest Feasible Descent as an initial search tool, followed by a use of the more efficient SQP method for 'local' refinement was demonstrated to be effective on both classical test functions and a parallel beam problem. Key conclusions are as follows:

- The combined method is shown to leverage the 'best' of the component methods for an improved result on some multi-modal responses, without consequence to other test surfaces used.
- The proposed method of SFD is shown to be effective as a constrained search tool, incorporating both deterministic and programmatic feasibility elements as well as a sensitivity-based search vector.
- The proposed method is shown to be more tolerant of a coarse design variable step size as a stopping criterion than SQP implemented individually. This is an important benefit with respect to use with EBBF's.
- The platform-style parallel beam structure was successfully optimized for harmonic loading, with significant improvements to peak response amplitudes, even though natural frequencies remained in the frequency range of interest due to design constraints of the system.

8. References

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