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A Sequential Optimization and Mixed Uncertainty Analysis Method Based on Taylor Series Approximation

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1. Abstract
In this paper reliability-based optimization (RBO) under both aleatory and epistemic uncertainties is studied based on combined probability and evidence theory. Traditionally the mixed uncertainty analysis is directly nested in optimization which is computationally prohibitive. To solve this problem, an effective way is to decompose the RBO problem into separate deterministic optimization and mixed uncertainty analysis sub-problems by sequential optimization and mixed uncertainty analysis (SOMUA) method, which are solved sequentially and alternately till convergence. SOMUA transforms the RBO problem into its quasi-equivalent deterministic formulation based on the inverse Most Probable Point (iMPP) of objective and constraint functions in each focal element. As the iMPP identification calculation is complex, the computational cost grows rapidly with the increase of focal elements. To improve the efficiency of SOMUA, in this paper it is proposed to use Taylor approximation to transform deterministic optimization. The efficacy of the proposed method is demonstrated with two test problems. It shows that the computational cost can be greatly reduced. However, the optimum may be very close to but not as good as that of SOMUA, which needs further research.

2. Keywords: Reliability-based optimization, Aleatory uncertainty, Epistemic uncertainty, Sequential optimization, Taylor series approximation

3. Introduction
Due the existence of uncertainties in engineering, reliability-based optimization (RBO) is widely studied and applied to enhance the system reliability. Generally uncertainties include two categories: the aleatory type arising from the inherent system randomness and the epistemic type due to subjective lack of knowledge [1]. In this paper, combined probability and evidence theory is used to deal with the mixed uncertainties. Based on probability and evidence theory, the RBO problem under mixed uncertainties is formulated as

$$\begin{cases} \text{find} & \boldsymbol{\mu}_{\mathbf{x}} \\ \min & \tilde{f} \\ \text{s.t.} & \text{Pl}\{f(\mathbf{x},\mathbf{p},\mathbf{z}) > \tilde{f}\} \le P_{f\_\text{obj}} = 1 - R_{\text{obj}} \\ & \text{Pl}\{g(\mathbf{x},\mathbf{p},\mathbf{z}) < c\} \le P_{f\_\text{con}} = 1 - R_{\text{con}} \\ & \mathbf{x}^L \le \boldsymbol{\mu}_{\mathbf{x}} \le \mathbf{x}^U \end{cases} \tag{1}$$

where P_{f_obj} and P_{f_con} are the target failure plausibility of the objective and constraint respectively. The design variable vector $\mathbf{x}$ is subject to random uncertainties and its mean value is optimized. The parameter vector $\mathbf{p}$ and $\mathbf{z}$ are random and epistemic uncertainty vectors respectively. From this formulation it is clear that at each search point the failure plausibility of the objective and constraint must be analysed which needs conduct the expensive mixed uncertainty analysis. Thus if the mixed uncertainty analysis is directly nested in the optimization, the computational cost would be unaffordable. To alleviate this problem, Yao proposed a sequential optimization and mixed uncertainty analysis (SOMUA) method to decompose the RBO problem into separate deterministic optimization and mixed uncertainty analysis sub-problems, which are solved sequentially and alternately until convergence is achieved [2]. SOMUA firstly decomposes the total reliability target into each focal element of the epistemic uncertainties. Then in each focal element the uncertain objective and constraint are transformed into the quasi-equivalent deterministic formulations by calculating the inverse Most Probable Point (iMPP) corresponding to the target reliability target in this focal element. The iMPP search is based on optimization, which induces extra calculation and gets worse when the number of focal elements grows big. To solve this problem, it is proposed to use Taylor approximation to transform deterministic optimization. In each cycle, uncertainty analysis is conducted at the deterministic optimum to obtain its MPP in each focal element. Then the epistemic uncertainties are assigned the values of MPP under the assumption that the worst case will happen with these values. Since the epistemic uncertainties are fixed, only random uncertainties are left. Thus the uncertain distributions of the objective and constraint functions are also random, which can be approximated by Taylor series approximation methods. As this method has the same structure as SOMUA to conduct the optimization and mixed uncertainty analysis sequentially, it is named as Taylor-SOMUA indicating that the deterministic transformation is based on Taylor series. The original SOMUA method in is denoted as iMPP-SOMUA emphasizing the iMPP based deterministic transformation. The rest of the paper is structured as follows. First, the preliminaries of RBO and the SOMUA method are introduced. Then the proposed method Taylor-SOMUA is developed, followed by two test demonstrations. Finally some conclusion remarks are presented.

4. Preliminaries

4.1. Reliability analysis under mixed uncertainties

The probability space of a random uncertain vector $\mathbf{x} = [x_1\ x_2\ \cdots\ x_{N_x}]$ is described by a triple $(X, \Upsilon, \Pr)$, where X is the universal set of all possible values of $\mathbf{x}$, Υ is a σ-algebra over X, and $\Pr$ is a probability measure indicating the probability that the elements of Υ occur. The evidence space of an epistemic uncertain vector $\mathbf{z} = (z_1\ z_2\ \cdots\ y_{N_z})$ is also described by a triple (C, Υ, m), where C contains all the possible distinct value set of $\mathbf{z}$, m is the basic probability assignment (BPA) function which maps C to [0,1] satisfying the following axioms: 1) $\forall A \in C,\quad m(A) \ge 0$; 2) for the empty set $\varnothing$, $m(\varnothing) = 0$; 3) for all the $A \in C$, $\sum m(A) = 1$. The set A which satisfies $m(A) > 0$ is called a focal element. Υ is the set of all the focal elements [3, 4].

For a system response function $g(\mathbf{x},\mathbf{z})$, its input includes both the aleatory uncertain vector $\mathbf{x}$ defined by $(\Omega, \mathcal{A}, \Pr)$ and the epistemic uncertain vector $\mathbf{z}$ described by (C, Υ, m) with N_C focal elements. Denote the failure region as $F = \{(\mathbf{x},\mathbf{z}) \mid g(\mathbf{x},\mathbf{z}) < a\}$. The precise probability of failure is bounded by its lower limit called belief (Bel) and its up limit called plausibility (Pl) defined as [2, 5, 6]

$$\mathrm{Bel}\{(\mathbf{x},\mathbf{z}) \in F\} = \sum_{k=1}^{N_C} \left(m(c_k) \cdot \mathrm{Bel}_k\{(\mathbf{x},\mathbf{z}) \in F\} \right); \quad \mathrm{Bel}_k\{(\mathbf{x},\mathbf{z}) \in F\} = \Pr\{\mathbf{x} \mid \forall \mathbf{z} \in c_k,\ g(\mathbf{x},\mathbf{z}) < a\} \tag{2}$$

$$\mathrm{Pl}\{(\mathbf{x},\mathbf{z}) \in F\} = \sum_{k=1}^{N_C} \left(m(c_k) \cdot \mathrm{Pl}_k\{(\mathbf{x},\mathbf{z}) \in F\} \right); \quad \mathrm{Pl}_k\{(\mathbf{x},\mathbf{z}) \in F\} = \Pr\{\mathbf{x} \mid \exists \mathbf{z} \in c_k,\ g(\mathbf{x},\mathbf{z}) < a\} \tag{3}$$

$\mathrm{Bel}_k\{(\mathbf{x},\mathbf{z}) \in F\}$ and $\mathrm{Pl}_k\{(\mathbf{x},\mathbf{z}) \in F\}$ are called the sub-belief and sub-plausibility of the focal element $c_k (1 \le k \le N_C)$. The methods to calculate $\mathrm{Bel}_k\{(\mathbf{x},\mathbf{z}) \in F\}$ and $\mathrm{Pl}_k\{(\mathbf{x},\mathbf{z}) \in F\}$ are referred to [7] and [5].

4.2. The SOMUA method

Denote the cycle number as $i = 1$. Directly ignore all the uncertainties (the uncertain variables are assigned with fixed values) and run the deterministic optimization. Denote the optimum as $\mu_{\mathbf{x}}^{(i)*}$ and its objective response as $\tilde{f}^{(i)*}$. Analyze the plausibility of the objective failure and the constraint failure at the optimum $\mu_{\mathbf{x}}^{(i)*}$ under mixed uncertainties with the mixed uncertainty analysis method SLO-FORM-UUA [5]. For each focal element $c_k (1 \le k \le N_C)$, the sub-plausibility $\mathrm{Pl}_k^{(i)}(F_{\mathrm{obj}})$ and $\mathrm{Pl}_k^{(i)}(F_{\mathrm{con}})$ can be first calculated with the MPP $[\mathbf{x}_{k\_\mathrm{MPP\_obj}}^{(i)*}, \mathbf{p}_{k\_\mathrm{MPP\_obj}}^{(i)*}, \mathbf{z}_{k\_\mathrm{MPP\_obj}}^{(i)*}]$ and $[\mathbf{x}_{k\_\mathrm{MPP\_con}}^{(i)*}, \mathbf{p}_{k\_\mathrm{MPP\_con}}^{(i)*}, \mathbf{z}_{k\_\mathrm{MPP\_con}}^{(i)*}]$. Then the total failure plausibility $\mathrm{Pl}^{(i)}(F_{\mathrm{obj}})$ and $\mathrm{Pl}^{(i)}(F_{\mathrm{con}})$ can be calculated with (3). Calculate the target sub-plausibility $\mathrm{Pl}_{k\_T}^{(i+1)}(F_{\mathrm{obj}})$ and $\mathrm{Pl}_{k\_T}^{(i+1)}(F_{\mathrm{con}})$ of the objective failure and the constraint failure for each focal element $c_k (1 \le k \le N_C)$ by $\mathrm{Pl}_{k\_T}^{(i+1)}(F_{\mathrm{obj}}) = \mathrm{Pl}_k^{(i)}(F_{\mathrm{obj}}) - \Delta\mathrm{Pl}_{\mathrm{obj}}^{(i)}$ and $\mathrm{Pl}_{k\_T}^{(i+1)}(F_{\mathrm{con}}) = \mathrm{Pl}_k^{(i)}(F_{\mathrm{con}}) - \Delta\mathrm{Pl}_{\mathrm{con}}^{(i)}$ where $\Delta\mathrm{Pl}_{\mathrm{obj}}^{(i)} = \mathrm{Pl}^{(i)}(F_{\mathrm{obj}}) - P_{f\_\mathrm{obj}}$ and $\Delta\mathrm{Pl}_{\mathrm{con}}^{(i)} = \mathrm{Pl}^{(i)}(F_{\mathrm{con}}) - P_{f\_\mathrm{con}}$. Identify the corresponding inverse MPP of $\mu_{\mathbf{x}}^{(i)*}$, which are denoted as $[\mathbf{x}_{k\_\mathrm{iMPP\_obj}}^{(i)*}, \mathbf{p}_{k\_\mathrm{iMPP\_obj}}^{(i)*}, \mathbf{z}_{k\_\mathrm{iMPP\_obj}}^{(i)*}]$ and $[\mathbf{x}_{k\_\mathrm{iMPP\_con}}^{(i)*}, \mathbf{p}_{k\_\mathrm{iMPP\_con}}^{(i)*}, \mathbf{z}_{k\_\mathrm{iMPP\_con}}^{(i)*}]$ for the objective and constraint in k th focal element respectively. Then the deterministic optimization problem for the $i+1$ th cycle can be formulated as

$$\begin{cases} \text{find} & \boldsymbol{\mu}_{\mathbf{x}}^{(i+1)} \\ \min & \tilde{f}^{(i+1)} = \max\limits_{1 \le k \le N_C} f_k^{(i+1)}(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)}), \quad f_k^{(i+1)}(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)}) = f\left(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)} - \mathbf{s}_{k\_\mathrm{obj}}^{(i+1)}, \mathbf{p}_{k\_\mathrm{iMPP\_obj}}^{(i)*}, \mathbf{z}_{k\_\mathrm{iMPP\_obj}}^{(i)*} \right) \\ \text{s.t.} & g_k^{(i+1)}(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)}) \ge c, \quad 1 \le k \le N_C \\ & g_k^{(i+1)}(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)}) = g\left(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)} - \mathbf{s}_{k\_\mathrm{con}}^{(i+1)}, \mathbf{p}_{k\_\mathrm{iMPP\_con}}^{(i)*}, \mathbf{z}_{k\_\mathrm{iMPP\_con}}^{(i)*} \right) \\ & \mathbf{s}_{k\_\mathrm{obj}}^{(i+1)} = \boldsymbol{\mu}_{\mathbf{x}}^{(i)*} - \mathbf{x}_{k\_\mathrm{iMPP\_obj}}^{(i)*}, \quad \mathbf{s}_{k\_\mathrm{con}}^{(i+1)} = \boldsymbol{\mu}_{\mathbf{x}}^{(i)*} - \mathbf{x}_{k\_\mathrm{iMPP\_con}}^{(i)*} \\ & \mathbf{x}^L \le \boldsymbol{\mu}_{\mathbf{x}}^{(i+1)} \le \mathbf{x}^U \end{cases} \tag{4}$$

Increase the cycle number $i = i+1$ and conduct the deterministic optimization of the next cycle. Repeat the preceding steps until convergence is achieved. For more detailed introduction of SOMUA, please refer to [2].

5. Taylor-SOMUA

5.1. The transformation of deterministic objective and constraint

At the optimum of the i th deterministic optimization, conduct the mixed uncertainty analysis and obtain the MPP $[\mathbf{x}_{k\_\mathrm{MPP\_con}}^{(i)*}, \mathbf{p}_{k\_\mathrm{MPP\_con}}^{(i)*}, \mathbf{z}_{k\_\mathrm{MPP\_con}}^{(i)*}]$. Assign the fixed value $\mathbf{z}_{k\_\mathrm{MPP\_con}}^{(i)*}$ to the epistemic uncertainty vector $\mathbf{z}$ under assumption that the failure will occur with bigger probability when the epistemic uncertainty vector is at this value. Then only random uncertain vectors $\mathbf{x}$ and $\mathbf{p}$ are left. Accordingly the constraint function response $g(\mathbf{x}, \mathbf{p}, \mathbf{z}_{k\_\mathrm{MPP\_con}}^{(i)*})$ is also random. Denote $g_{k\_rand}(\mathbf{x}, \mathbf{p}) = g(\mathbf{x}, \mathbf{p}, \mathbf{z}_{k\_\mathrm{MPP\_con}}^{(i)*})$. Assume the distribution of g_{k_rand} follows the normal distribution, then its mean and standard deviation can be approximated based on the first order Taylor series expansion as follows:

$$\begin{aligned} &\mu(g_{k\_rand}) = g_{k\_rand}(\boldsymbol{\mu}_{\mathbf{x}}, \boldsymbol{\mu}_{\mathbf{p}}) \\ &\sigma(g_{k\_rand}) = \sqrt{\sum_{i=1}^{n_X} \left(\frac{\partial g_{k\_rand}(\mathbf{x}, \mathbf{p})}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \sum_{i=1}^{n_P} \left(\frac{\partial g_{k\_rand}(\mathbf{x}, \mathbf{p})}{\partial p_i} \right)^2 \sigma_{p_i}^2} \end{aligned} \tag{5}$$

According to the target failure plausibility $\beta_{k\_T\_\mathrm{con}}^{(i+1)} = \Phi^{-1}(\mathrm{Pl}_{k\_T}^{(i+1)}(F_{\mathrm{con}}))$, where Φ is standard normal cumulative distribution function and generally $\beta_{k\_T\_\mathrm{con}}^{(i+1)} < 0$ as $\mathrm{Pl}_{k\_T}^{(i+1)}(F_{\mathrm{con}}) < 0.5$, reliability constraint can be formulated as

$$\mu(g_{k\_rand}(\mathbf{x}, \mathbf{p})) + \beta_{k\_T\_\mathrm{con}}^{(i+1)} \bullet \sigma(g_{k\_rand}(\mathbf{x}, \mathbf{p})) > c \tag{6}$$

Actually $g_{k\_rand}(\mathbf{x}, \mathbf{p})$ may not be normally distributed, thus it should be verified whether the normal distribution assumption is rational and the accuracy of approximation formulation (6) is within acceptable level. Denote $\beta_{k\_\mathrm{con}}^{(i)} = \Phi^{-1}(\mathrm{Pl}_k^{(i)}(F_{\mathrm{con}}))$. Substitute $[\mathbf{x}_{k\_\mathrm{MPP\_con}}^{(i)*}, \mathbf{p}_{k\_\mathrm{MPP\_con}}^{(i)*}]$ into (5) and obtain $\mu_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand})$ and $\sigma_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand})$. If g_{k_rand} follows normal distribution, then the equation $\mu_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand}) + \beta_{k\_\mathrm{con}}^{(i)} \sigma_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand}) = c$ should exist. Thus denote $\varepsilon = \left| \mu_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand}) + \beta_{k\_\mathrm{con}}^{(i)} \sigma_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand}) - c \right|$. If ε is smaller than the predefined threshold, then the normal distribution assumption is rational. Otherwise it is suggested to revise the standard deviation as

$$\sigma_{k\_\mathrm{con}}^{(i+1)}(g_{k\_rand}) = \left(c - \mu_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand}) \right) / \beta_{k\_\mathrm{con}}^{(i)} \tag{7}$$

And use this value instead of the one estimated in (5) for constraint transformation in (6). The transformation of the objective function is the same as the aforementioned procedure, and the deterministic objective is formulated as

$$\min \quad \tilde{f} = \max_{1 \le k \le N_C} \left[\begin{aligned} &f(\boldsymbol{\mu}_{\mathbf{x}}, \boldsymbol{\mu}_{\mathbf{p}}, \mathbf{z}_{k\_\mathrm{MPP\_obj}}^{(i)*}) \\ &-\Phi^{-1}(\mathrm{Pl}_{k\_T}^{(i+1)}(F_{\mathrm{obj}})) \bullet \sqrt{\sum_{i=1}^{n_X} \left(\frac{\partial f(\mathbf{x}, \mathbf{p}, \mathbf{z}_{k\_\mathrm{MPP\_obj}}^{(i)*})}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \sum_{i=1}^{n_P} \left(\frac{\partial f(\mathbf{x}, \mathbf{p}, \mathbf{z}_{k\_\mathrm{MPP\_obj}}^{(i)*})}{\partial p_i} \right)^2 \sigma_{P_i}^2} \end{aligned} \right] \tag{8}$$

5.2. The Taylor-SOMUA algorithm

To sum up, the detailed procedure of the Taylor-SOMUA algorithm is as follows:

Step 1: Denote $i = 1$. Directly ignore all the uncertainties and run the deterministic optimization.

Step 2: For the i th cycle solve the deterministic optimization problem. Denote the optimum as $\boldsymbol{\mu}_{\mathbf{X}}^{(i)*}$ and the objective value as $\tilde{f}^{(i)*}$.

Step 3: Conduct uncertainty analysis at $\boldsymbol{\mu}_{\mathbf{X}}^{(i)*}$ with SLO-FORM-UUA method [5]. For each focal element $c_k (1 \le k \le N_C)$, calculate the sub-plausibility $\mathrm{Pl}_k^{(i)}(F_{\mathrm{obj}})$ and $\mathrm{Pl}_k^{(i)}(F_{\mathrm{con}})$ with the MPP $[\mathbf{x}_{k\_\mathrm{MPP\_obj}}^{(i)*}, \mathbf{p}_{k\_\mathrm{MPP\_obj}}^{(i)*}, \mathbf{z}_{k\_\mathrm{MPP\_obj}}^{(i)*}]$ and $[\mathbf{x}_{k\_\mathrm{MPP\_con}}^{(i)*}, \mathbf{p}_{k\_\mathrm{MPP\_con}}^{(i)*}, \mathbf{z}_{k\_\mathrm{MPP\_con}}^{(i)*}]$. Denote the objective failure plausibility as $\mathrm{Pl}^{(i)}(F_{\mathrm{obj}})$ and constraint failure plausibility as $\mathrm{Pl}^{(i)}(F_{\mathrm{con}})$. If all the failure plausibility satisfies the reliability requirement, go to Step 5. Otherwise go to next step.

Step 4: Fix the value of epistemic uncertainty vector as $\mathbf{z}_{k\_\mathrm{MPP\_con}}^{(i)*}$ or $\mathbf{z}_{k\_\mathrm{MPP\_obj}}^{(i)*}$ and substitute it into the constraint and objective function. Denote $g_{k\_rand}(\mathbf{x}, \mathbf{p}) = g(\mathbf{x}, \mathbf{p}, \mathbf{z}_{k\_\mathrm{MPP\_con}}^{(i)*})$ and $f_{k\_rand}(\mathbf{x}, \mathbf{p}) = f(\mathbf{x}, \mathbf{p}, \mathbf{z}_{k\_\mathrm{MPP\_obj}}^{(i)*})$. Denote $\beta_{k\_\mathrm{con}}^{(i)} = \Phi^{-1}(\mathrm{Pl}_k^{(i)}(F_{\mathrm{con}}))$ and $\beta_{k\_\mathrm{obj}}^{(i)} = \Phi^{-1}(\mathrm{Pl}_k^{(i)}(F_{\mathrm{obj}}))$. Denote $\varepsilon_{k\_\mathrm{con}}^{(i)} = \left| \mu_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand}) + \beta_{k\_\mathrm{con}}^{(i)} \sigma_{k\_\mathrm{con}}^{(i)*}(g_{k\_rand}) - c \right|$

and $\varepsilon_{k\_\mathrm{obj}}^{(i)}=\left|\mu_{k\_\mathrm{obj}}^{(i)*}(f_{k\_rand})-\beta_{k\_\mathrm{obj}}^{(i)}\sigma_{k\_\mathrm{obj}}^{(i)*}(f_{k\_rand})-\tilde{f}^{(i)*}\right|$. Then the deterministic optimization problem of $i+1$ th cycle is formulated as

$$\begin{cases}
\text{find} & \boldsymbol{\mu}_{\mathbf{x}}^{(i+1)} \quad (\mathbf{x}^L \le \boldsymbol{\mu}_{\mathbf{x}}^{(i+1)} \le \mathbf{x}^U) \\
\text{min} & \tilde{f}^{(i+1)} = \max\limits_{1\le k\le N_C} f_k^{(i+1)}(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)}) \\
 & f_k^{(i+1)}(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)}) = f(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)},\boldsymbol{\mu}_{\mathbf{p}},\mathbf{z}_{k_\mathrm{MPP_obj}}^{(i)*}) - \Phi^{-1}(\mathrm{Pl}_{k_T}^{(i+1)}(F_{\mathrm{obj}}))\bullet\sigma_{k_\mathrm{obj}}^{(i+1)}(f_{k_rand}) \\
 & \text{if} \quad \varepsilon_{k_\mathrm{obj}}^{(i)} > \varepsilon_0, \quad \sigma_{k_\mathrm{obj}}^{(i+1)}(f_{k_rand}) = \left(c-\mu_{k_\mathrm{obj}}^{(i)*}(f_J)\right)/\beta_{k_\mathrm{obj}}^{(i)} \\
 & \text{if} \quad \varepsilon_{k_\mathrm{obj}}^{(i)} \le \varepsilon_0, \quad \sigma_{k_\mathrm{obj}}^{(i+1)}(f_{k_rand}) = \sqrt{\sum_{i=1}^{n_X}\left(\dfrac{\partial f(\mathbf{x},\mathbf{p},\mathbf{z}_{k_\mathrm{MPP_obj}}^{(i)*})}{\partial X_i}\right)^2\sigma_{x_i}^2+\sum_{i=1}^{n_P}\left(\dfrac{\partial f(\mathbf{x},\mathbf{p},\mathbf{z}_{k_\mathrm{MPP_obj}}^{(i)*})}{\partial P_i}\right)^2\sigma_{P_i}^2} \\
\text{s.t.} & g_k^{(i+1)}(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)}) \ge c, \quad 1\le k\le N_C \\
 & g_k^{(i+1)}(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)}) = g(\boldsymbol{\mu}_{\mathbf{x}}^{(i+1)},\boldsymbol{\mu}_{\mathbf{p}},\mathbf{z}_{k_\mathrm{MPP_con}}^{(i)*}) + \Phi^{-1}(\mathrm{Pl}_{k_T}^{(i+1)}(F_{\mathrm{con}}))\bullet\sigma_{k_\mathrm{con}}^{(i+1)}(g_{k_rand}) \\
 & \text{if} \quad \varepsilon_{k_\mathrm{con}}^{(i)} > \varepsilon_0, \quad \sigma_{k_\mathrm{con}}^{(i+1)}(g_{k_rand}) = \left(c-\mu_{k_\mathrm{con}}^{(i)*}(g_{k_rand})\right)/\beta_{k_\mathrm{con}}^{(i)} \\
 & \text{if} \quad \varepsilon_{k_\mathrm{con}}^{(i)} \le \varepsilon_0, \quad \sigma_{k_\mathrm{con}}^{(i+1)}(g_{k_rand}) = \sqrt{\sum_{i=1}^{n_X}\left(\dfrac{\partial g(\mathbf{x},\mathbf{p},\mathbf{z}_{k_\mathrm{MPP_con}}^{(i)*})}{\partial x_i}\right)^2\sigma_{x_i}^2+\sum_{i=1}^{n_P}\left(\dfrac{\partial g(\mathbf{x},\mathbf{p},\mathbf{z}_{k_\mathrm{MPP_con}}^{(i)*})}{\partial p_i}\right)^2\sigma_{P_i}^2}
\end{cases} \tag{9}$$

where the symbol k in the subscript represents the focal element index. Denote $i=i+1$ and go to Step 2.

Step 5: Check convergence. If the relative change between the optimums of two consecutive cycles is smaller than the threshold, end the algorithm. Otherwise go to Step 4.

6. Tests

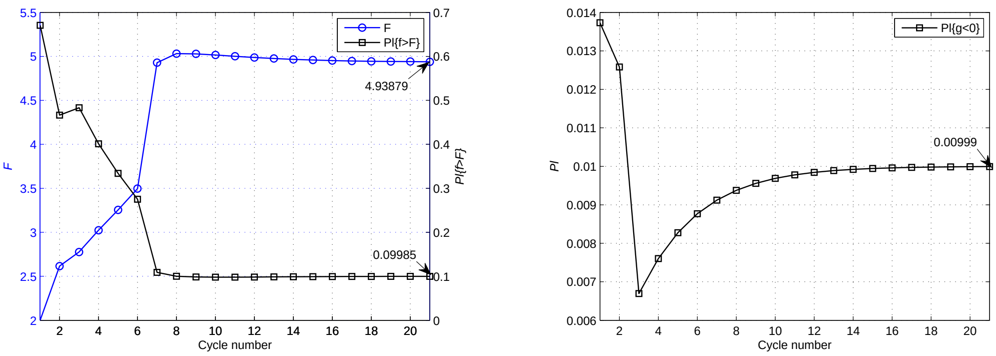
6.1. Test 1: a numerical problem

$$\begin{cases}
\text{find} & \mu_x \\
\text{min} & \tilde{f} \\
\text{s.t.} & \mathrm{Pl}\{f(x,p,z)>\tilde{f}\}\le 0.1, \quad f(x,p,z)=(x+2.5)^2+p+z \\
 & \mathrm{Pl}\{g(x,p,z)<0\}\le 0.01, \quad g(x,p,z)=-x+(z-0.7)+p, \quad -3\le\mu_x\le 2
\end{cases} \tag{10}$$

The optimization variable x is subject to normal distribution $N(\mu_x,1.0)$. The uncertain parameter p follows normal distribution $N(2.0,1.0)$. The BPA of the epistemic uncertainty z is as follows:

$$c_1=[-1,0), \quad m(c_1)=0.5; \quad c_2=[0,1], \quad m(c_2)=0.5 \tag{11}$$

The optimization result of Taylor-SOMUA is compared with that of iMPP-SOMUA and the traditional nested method, which are presented in Table 1. The convergence history is depicted in Figure 1. All of the three methods obtain optimization designs which satisfy reliability requirements. It can be observed that the optimum of Taylor-SOMUA is slightly bigger than that of other two methods, but its computational cost is the smallest, which proves its efficacy in balancing the computation cost and optimization effect.



(a) The convergence history of the objective and its failure plausibility

(b) The convergence history of the constraint failure plausibility

Figure 1: The optimization convergence history of Test 1

Table 1: The optimization results of Test 1

	iMPP-SOMUA	Taylor-SOMUA	Traditional nested method
x	-2.60076	-2.70490	-2.60073
$\mathrm{Pl}(F_{\mathrm{con}})$	0.00998	0.00956	0.01001
F	4.93895	5.02799	4.93891
$\mathrm{Pl}(F_{\mathrm{obj}})$	0.09994	0.09856	0.09997
Cycle number	9	9	1
Total number of function calls	8262	7985	169724
Number of function calls used for deterministic transformation	780	268	--

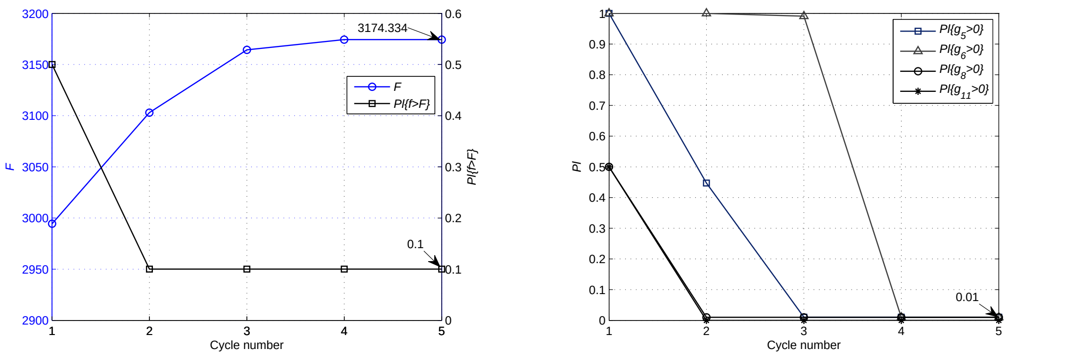
6.2. Test 2: The Golinski's speed reducer design optimization problem

$$\begin{cases} \text{find}: & \mu_{\mathbf{x}} = [\mu_{x_1} \ \ \mu_{x_2} \ \ x_3 \ \ \mu_{x_4} \ \ \mu_{x_5} \ \ \mu_{x_6} \ \ \mu_{x_7}]^T \\ \text{min}: & \tilde{f} \\ \text{s.t.} & \mathrm{Pl}\{f > \tilde{f}\} \le 10\%, \quad \mathrm{Pl}\{g_i > 0\} \le 1\%, \quad 1 \le i \le 11 \\ & f(\mathbf{x}) = 0.7854 x_1 x_2^2 (3.3333 x_3^2 + 14.9334 x_3 - 43.0934) \\ & \qquad -1.5079 x_1 (x_6^2 + x_7^2) + 7.477 (x_6^3 + x_7^3) + 0.7854 (x_4 x_6^2 + x_5 x_7^2) \\ & g_1 : 27.0/(x_1 x_2^2 x_3) - 1 \le 0, \quad g_2 : 397.5/(x_1 x_2^2 x_3^2) - 1 \le 0, \quad g_3 : 1.93 x_4^3/(x_2 x_3 x_6^4) - 1 \le 0 \\ & g_4 : 1.93 x_5^3/(x_2 x_3 x_7^4) - 1 \le 0, \quad g_5 : A_1/B_1 - 1100 \le 0, \quad g_6 : A_2/B_2 - 850 \le 0 \\ & g_7 : x_2 x_3 - 40.0 \le 0, \quad g_8 : 5.0 \le x_1/x_2, \quad g_9 : x_1/x_2 \le 12.0 \\ & g_{10} : (1.5 x_6 + 1.9)/x_4 - 1 \le 0, \quad g_{11} : (1.1 x_7 + 1.9)/x_5 - 1 \le 0 \\ & A_1 = \left[\left(\dfrac{a_1 x_4}{x_2 x_3}\right)^2 + a_2 \times 10^6\right]^{0.5}, B_1 = a_3 x_6^3, A_2 = \left[\left(\dfrac{a_1 x_5}{x_2 x_3}\right)^2 + a_4 \times 10^6\right]^{0.5}, B_2 = a_3 x_7^3 \\ & 2.6 \le \mu_{x_1} \le 3.6, 0.7 \le \mu_{x_2} \le 0.8, 17 \le x_3 \le 28, 7.3 \le \mu_{x_4} \le 8.3 \\ & 7.3 \le \mu_{x_5} \le 8.3, 2.9 \le \mu_{x_6} \le 3.9, 5.0 \le \mu_{x_7} \le 5.5 \end{cases} \tag{12}$$

The optimization variables except x_3 are random. The mean values are optimized and the standard deviations are [21um, 1um, 30um, 30um, 21um, 30um] for $[x_1 \ \ x_2 \ \ x_4 \ \ x_5 \ \ x_6 \ \ x_7]$ respectively. For the four epistemic uncertainties a_1, a_2, a_3, and a_4, one interval is considered for each uncertainty as follows:

$$a_1 \in [740.0, 750.0], a_2 \in [16.5, 17.5], a_3 \in [0.09, 0.11], a_4 \in [157, 158] \tag{13}$$

The optimization results of Taylor-SOMUA and iMPP-SOMUA are presented in Table 2. The convergence history of Taylor-SOMUA is depicted in Figure 2. The optimum of Taylor-SOMUA is slightly bigger than that of iMPP-SOMUA, but its computational cost is much less than iMPP-SOMUA, which proves its efficiency.



(a) The convergence history of the objective and its failure plausibility (b) The convergence history of the failure plausibility of the hard constraints

Figure 2: The optimization convergence history of Test 2

Table 2: The optimization results of Test 2

		Deterministic optimum	Taylor-SOMUA optimum	iMPP-SOMUA optimum
Design variables		3.5, 0.7, 17, 7.3, 7.7153, 3.3502, 5.2867	3.50502, 0.7, 7.3, 7.94452, 3.49492, 5.48559	3.5050, 0.7, 17, 7.3, 7.9348, 3.4949, 5.4856
Objective		2994.355	3174.334	3174.108
Constraints	Pl{f >F}	0.5	0.1	0.1
	Pl{g5>0}	1	0.01	0.01
	Pl{g6>0}	1	0.01	0.01
	Pl{g8>0}	0.5	0.01	0.01
	Pl{g11>0}	0.5	0	0
Cycle number		--	6	5
Total number of function calls		--	875	3290

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8. Conclusions
In this paper, the RBO method under mixed uncertainties is studied based on sequential optimization and mixed uncertainty analysis method. To alleviate the computational problem of the original SOMUA method which needs iMPP to transform the deterministic optimization formulation, it is proposed to use Taylor approximation to transform the deterministic objective and constraint in this paper. The efficacy of the proposed method is demonstrated with two test problems. It shows that the computational cost can be greatly reduced. However, the optimum may be very close to but not as good as that of iMPP-SOMUA, which proves the efficacy of the proposed method in balancing the computational efficiency and optimization effect. However, the applicability of this method in highly nonlinear optimization problems still needs further studies.

9. References
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