Elastic Moduli Identification Method for Orthotropic Structures based on Vibration Data

Akira Saito[1], Yasunari Nishikawa[2], Shintaro Yamasaki[2], Kikuo Fujita[2], Atsushi Kawamoto[1], Masakatsu Kuroishi[1], Hideo Nakai[1]

[1] Toyota Central R&D Labs., Inc. Nagakute, Japan, akira-saito@mosk.tytlabs.co.jp
[2] Department of Mechanical Engineering, Osaka University, Suita, Japan, yamasaki@mech.eng.osaka-u.ac.jp

1. Abstract
A novel numerical-experimental methodology for the identification of elastic moduli of orthotropic structures is proposed. Special attention is given to the elastic moduli of laminated electrical steel sheets, which are widely used for the magnetic cores of electric motors and generators. The elastic moduli are determined specifically for use with finite element vibration analyses, such that the dynamic characteristics of such structures can properly be predicted by using the identified elastic moduli. The identification problem is formulated as an inverse problem with nonlinear least squares fit between the measured and computed modal frequencies. The problem is sequentially solved with increasing number of modes that are carefully yet automatically selected based on the analytic sensitivity of the modal frequencies on the elastic moduli. Using the results of numerical experiments, it is shown that the optimal solution obtained by the proposed method converges to the accurate elastic moduli as the number of modes increases. Furthermore, it is also shown that the method not only converges faster but also is numerically more stable than conventional methods. Finally, the method is applied to the experimentally-obtained modal frequencies of the laminated electrical steel sheets, and successfully identifies the elastic moduli where the finite element modal analysis can reproduce accurate modal frequencies.

2. Keywords: Inverse problem, Nonlinear least squares, Orthotropic material, Electrical steel sheets, Laminated plates.

3. Introduction
There is a growing demand to model and predict the dynamic characteristics of the electric machines, as electrified vehicles such as hybrid-electric vehicles and electric vehicles become prevalent. One of the most important yet challenging tasks is the accurate modeling of the modal characteristics of the electric machines. Without the accurate modal characteristics, even if the magnetic force excitation is accurately predicted, it is impossible to accurately capture the forced response of the electric machines [1]. In particular, we shall pay special attention to the modal characteristics of the laminated electrical steel sheets, which are widely used as the magnetic cores of the electric machines. The elastic behavior of the laminated electrical steel is represented by orthotropic constitutive relationship between stress and strain. In this paper, a novel elastic moduli identification method for orthotropic structures is proposed, which is based on the nonlinear least squares (LS) parameter identification procedure using measured mode shapes and frequencies, in conjunction with finite element (FE) model updating technique. The proposed method incorporates the LS procedure with successive increments of the number of modal frequencies and initial condition updates, which enables fast and accurate convergence of the identification procedure.
This paper is organized as follows. Mathematical formulation of the nonlienar LS problem is described in Section 4. In Section 5, the validity of the proposed approach is confirmed by numerical experiments. In Section 6, the methods are applied to the measured mode shapes and frequencies of the laminated electrical steel sheets, and the effectiveness of the proposed methods is discussed. Conclusions are provided in Section 7.

4. Mathematical formulation
4.1 Nonlinear least squares minimization
Let us suppose that the space occupied by the vibrating body of interest is denoted as $\Omega \subset \mathbb{R}^3$, and the body of interest consists of a material with orthotropic constitutive relationship. Assuming that a set of n modal frequencies of the body has been extracted from the results of modal testing, they are denoted as $\tilde{f}_k$, $k = 1, \dots, n$. With the measured modal frequencies, we try to identify the set of nine independent engineering elastic moduli of the orthotropic material, which is denoted here as $\boldsymbol{p} = [E_1, E_2, E_3, G_{12}, G_{23}, G_{13}, \nu_{12}, \nu_{23}, \nu_{13}]^{\mathrm{T}}$. Denoting numerically-obtained modal frequencies with $\boldsymbol{p}$ as $f_k(\boldsymbol{p})$, $k = 1, \dots, n$, we solve the following minization problem:

$$\underset{\boldsymbol{p}}{\text{minimize}}\, \mathscr{L}_n(\boldsymbol{p}) := \frac{1}{2}\|\boldsymbol{r}_n(\boldsymbol{p})\|^2, \tag{1}$$

where $\|\|$ denotes the 2-norm of a vector, $\boldsymbol{r}_n(\boldsymbol{p}) = \boldsymbol{f}_n(\boldsymbol{p}) - \tilde{\boldsymbol{f}}_n(\boldsymbol{p})$, $\boldsymbol{f}_n(\boldsymbol{p}) = [f_1(\boldsymbol{p}), f_2(\boldsymbol{p}), \ldots, f_n(\boldsymbol{p})]^{\mathrm{T}}$, and $\tilde{\boldsymbol{f}}_n = [\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n]^{\mathrm{T}}$. For the minimization algorithm, we assume general-purpose, gradient-based nonlinear minimization algorithm, such as Gauss-Newton methods and quasi-Newton methods. Fundamentally, we seek the stationary point of the function where the first order derivative of $\mathscr{L}_n(\boldsymbol{p})$ with respect to $\boldsymbol{p}$ vanishes, i.e.,

$$\nabla_{\boldsymbol{p}} \mathscr{L}_n(\boldsymbol{p}) = \boldsymbol{0}, \tag{2}$$

where $\nabla_{\boldsymbol{p}}$ denotes the vector differential operator with respect to $\boldsymbol{p}$. It means that the derivative of the eigenfrequencies with respect to $\boldsymbol{p}$ needs to be evaluated because

$$\nabla_{\boldsymbol{p}} \mathscr{L}_n(\boldsymbol{p}) = \boldsymbol{J}_n^{\mathrm{T}}(\boldsymbol{p}) \boldsymbol{r}_n(\boldsymbol{p}), \tag{3}$$

where $\boldsymbol{J}_n(\boldsymbol{p})$ is the Jacobian matrix whose ith row and jth column element is denoted as

$$[\boldsymbol{J}_n(\boldsymbol{p})]_{ij} = \frac{\partial f_i(\boldsymbol{p})}{\partial p_j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, 9. \tag{4}$$

The $i-j$ component of the Jacobian matrix can be written as:

$$[\boldsymbol{J}_n(\boldsymbol{p})]_{ij} = -\frac{1}{8\pi^2 f_i(\boldsymbol{p})} \int_\Omega \tilde{\varepsilon}(\phi_i) \cdot \tilde{\boldsymbol{C}} \cdot \left(\frac{\partial \tilde{\boldsymbol{S}}}{\partial p_j} \right) \cdot \tilde{\boldsymbol{C}} \cdot \tilde{\varepsilon}(\phi_i) \mathrm{d}\Omega. \tag{5}$$

$$\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{S}}^{-1} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{13} \end{bmatrix}^{-1}, \tag{6}$$

where ϕ_i denotes the ith mode shape, $\tilde{\varepsilon}$ denotes the Voigt notation of the Cauchy's strain tensor, $\tilde{C}$ and $\tilde{\boldsymbol{S}}$ denote the stiffness and the compliance matrices, respectively.

4.2 Successive augmentation of the least squares problem and initial condition updates

There are three key factors that greatly affect the results of the minimization problem: the number of modes, the initial conditions, and the types of modes used in the objective function. In this paper, we propose guidelines to deal with these factors to improve the accuracy of the solutions of the LS problem.

• Number of modes and initial conditions

In general, it is possible to improve the solution of the LS problem by using more measurement data than the number of paramters to be determined. In our case, if the number of modes exceeds the number of the elastic moduli, it becomes an over-determined problem, and the LS solution is expected to be improved. Namely, we augment the LS problem by adding more residuals to be minimized.

The initial condition for the minimization problem is another important factor, considering that this minimization problem is an ill-posed problem [2]. That is, a "good" initial condition that is close enough to the optimal solution should be selected a priori, before starting the minimization. Even though it is not always possible to choose a good initial condition, a simple alteration to the algorithm would alleviate this difficulty. Namely, during the successive augmentations of the LS problem, the initial values $\boldsymbol{p}_0$ for the minimization of $\mathscr{L}_k(\boldsymbol{p})$ are updated to the converged solution of $\mathscr{L}_{k-1}(\boldsymbol{p})$.

• Mode selection algorithm

When forming Eq. (1), it is not obvious which vibration modes should be used for the accurate determination of the elastic moduli. Therefore, we try to construct the mode selection criteria for Eq. (1) based on the following arguments. The Taylor expansion of the function $\mathscr{L}_k(\boldsymbol{p})$ around a specific point $\bar{\boldsymbol{p}}$ up to second order gives the following quadratic function:

$$\tilde{\mathscr{L}}_k(\boldsymbol{p}) := \mathscr{L}_k(\bar{\boldsymbol{p}}) + [\nabla_{\boldsymbol{p}} \mathscr{L}_k(\bar{\boldsymbol{p}})]^{\mathrm{T}} [\boldsymbol{p} - \bar{\boldsymbol{p}}] + \frac{1}{2} [\boldsymbol{p} - \bar{\boldsymbol{p}}]^{\mathrm{T}} \nabla_{\boldsymbol{p}}^2 \mathscr{L}_k(\boldsymbol{p}) [\boldsymbol{p} - \bar{\boldsymbol{p}}]. \tag{7}$$

where $\nabla_{\boldsymbol{p}}^2 \mathscr{L}_k(\boldsymbol{p})$ is the Hessian matrix that can also be written as

$$\nabla_{\boldsymbol{p}}^2 \mathscr{L}_k(\boldsymbol{p}) = \boldsymbol{J}_k^{\mathrm{T}} \boldsymbol{J}_k + \left(\nabla_{\boldsymbol{p}}^2 \boldsymbol{f}_k \right)^{\mathrm{T}} \boldsymbol{r}_k. \tag{8}$$

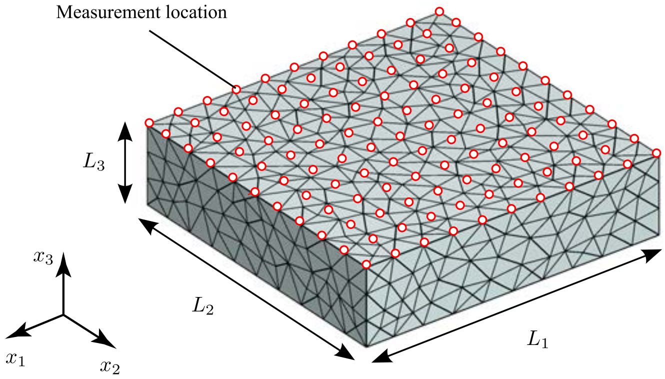


Figure 1: Numerical model with orthotropic material model

If we employed the exact Newton method, the algorithm would attempt to solve the minimization of Eq. (1) by successively finding the stationary point of the quadratic function $\tilde{\mathscr{L}}_k(\boldsymbol{p})$ at each iteration, i.e., $\boldsymbol{\nabla}_{\boldsymbol{p}}\tilde{\mathscr{L}}_k(\boldsymbol{p}) = \boldsymbol{0}$. It means that the increment $\Delta\boldsymbol{p}$ at each iteration would be determined by solving the following linear equations:

$$\left[\boldsymbol{J}_k^{\mathrm{T}}\boldsymbol{J}_k + \left(\boldsymbol{\nabla}^2_{\boldsymbol{p}}\boldsymbol{f}_k\right)^{\mathrm{T}}\boldsymbol{r}_k\right]\Delta\boldsymbol{p} = -\boldsymbol{\nabla}_{\boldsymbol{p}}\mathscr{L}_k(\boldsymbol{p}). \tag{9}$$

When the iteration reaches near the stationary point, $\boldsymbol{r}_k \approx \boldsymbol{0}$ and $\boldsymbol{\nabla}^2_{\boldsymbol{p}}\boldsymbol{r}_k$ can also be assumed to be very small. Therefore, if $\|\boldsymbol{r}_k\|_2$ is very small, or $\tilde{\mathscr{L}}_k(\boldsymbol{p})$ is evaluated near the stationary point, the Eq. (9) reduces to the following relationship:

$$\left(\boldsymbol{J}_k^{\mathrm{T}}\boldsymbol{J}_k\right)\Delta\boldsymbol{p} \approx -\boldsymbol{\nabla}_{\boldsymbol{p}}\mathscr{L}_k(\boldsymbol{p}). \tag{10}$$

One may notice that Eq. (10) implies that the accuracy of the increment $\Delta\boldsymbol{p}$ is influenced by the nature of the matrix $\boldsymbol{J}_k^{\mathrm{T}}\boldsymbol{J}_k$. That is, if the *condition number* of the matrix $\boldsymbol{J}_k^{\mathrm{T}}\boldsymbol{J}_k$, or equivalently that of $\boldsymbol{J}_k(\boldsymbol{p})$ is large, then the solution we obtain by solving Eq. (10) can be inaccurate. It means that it is important to keep the condition number of $\boldsymbol{J}_k^{\mathrm{T}}\boldsymbol{J}_k$ as small as possible, for the accurate and stable determination of the increment $\Delta\boldsymbol{p}$. Therefore, we propose that the mode sequence at each minimization be selected, such that the condition number of $\boldsymbol{J}_k$ is the smallest among the possible set of modes. Denoting the minimizer of $\mathscr{L}_k(\boldsymbol{p})$ as $\hat{\boldsymbol{p}}_k$, $\mathscr{L}_{k+1}(\boldsymbol{p})$ is updated such that:

$$\mathscr{L}_{k+1}(\boldsymbol{p}) = \mathscr{L}_k(\boldsymbol{p}) + \frac{1}{2}\left(f_\ell(\boldsymbol{p}) - \tilde{f}_\ell\right)^2, \tag{11}$$

where the mode index ℓ is found such that:

$$\kappa\left(\left[\boldsymbol{J}_k(\hat{\boldsymbol{p}}_k), \boldsymbol{\nabla}_{\boldsymbol{p}} f_\ell(\hat{\boldsymbol{p}}_k)\right]\right) = \text{minimum}, \tag{12}$$

where κ denotes the condition number, or the ratio between the maximum and the minimum singular values of the matrix. A more qualitative statement of this argument is that the algorithm chooses the modes so that the columns of $\boldsymbol{J}_k$ become as linearly-independent as possible.

5. Numerical validation: rectangular thick plate with orthotropic material model

In this section, the nonlinear LS problem presented in the previous section is examined using a numerical example. For the FE modal analyses as well as the evaluation of the eigenvalue sensitivity, COMSOL Multiphysics®was used. The BFGS algorithm [3] was employed for the minimization of Eq. (1), and implemented in the Matlab® environment. The numerical model is presented in Fig. 1, where $L_1 = 0.2$[m], $L_2 = 0.2$[m], and $L_3 = 0.05$[m]. The material model is an artificial orthotropic material with E_1=150GPa, E_2=180GPa, E_3=100GPa, G_{12}=60GPa, G_{23}=70GPa, G_{13}=80GPa, ν_{12}=0.1, ν_{23}=0.2, and ν_{13}=0.3. The analysis procedure is stated as follows. First, the modal analysis was conducted by an FEA, and the eigenvalues and the associated eigenvectors were exctracted. These eigenfrequencies are herein treated as the "measured" modal frequencies from a (numerical) experiment, and used as $\tilde{f}_k$, $k = 1, \ldots, 20$. Second, the eigenvectors were sampled at the evenly-spaced points on the top surface of the model as shown in Fig. 1. This simulates the vibration measurement of the mode shapes with a limited number of measurement locations. The Eq. (1) was then minimized using the "measured" data.

To demonstrate the advantage of the proposed algorithm, the minimization without the initial condition updates and the mode selection was also conducted, and the results are compared with those obtained by the proposed

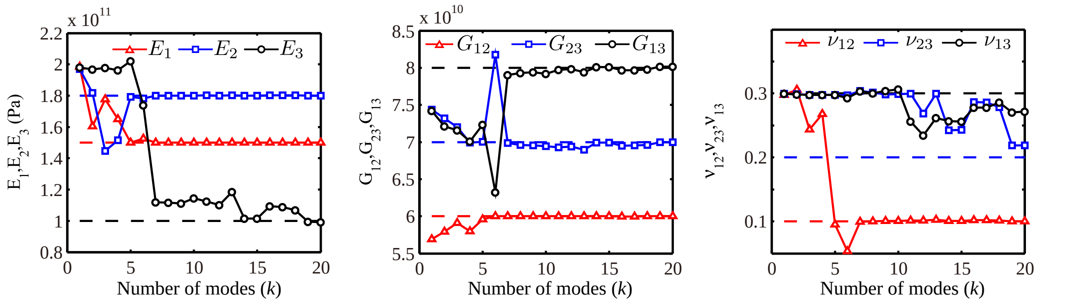


Figure 2: Convergence histories of the parameters (without initial condition updates and mode selection)

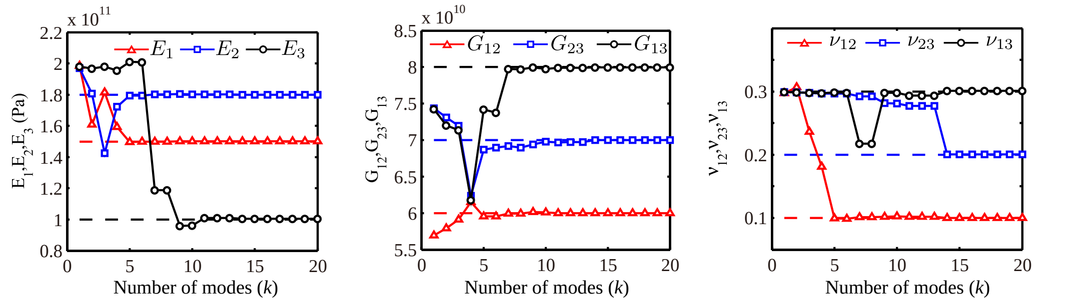


Figure 3: Convergence histories of the parameters (with initial condition updates and mode selection)

algorithm. The initial values of $\boldsymbol{p}$ for both algorithms were assumed to be the elastic moduli for an isotropic material, where E_1=200GPa, E_2=200GPa, E_3=200GPa, G_{12}=76.9GPa, G_{23}=76.9GPa, G_{13}=76.9GPa, ν_{12}=0.3, ν_{23}=0.3, and ν_{13}=0.3. The convergence histories of the elastic moduli without and with the initial condition updates and mode selection algorithm are shown respectively in Figs. 2 and 3. The dashed-lines in the figures show the target values for each modulus. As can be seen in Figs. 2 and 3, the converged solutions are improved as the number of modes increases for both cases. However, as seen in Fig. 2, the convergence histories fluctuates as the number of modes changes, and the algorithm even fails to identify the target values for ν_{23}, and ν_{13}. On the contrary, the proposed method successfully determines all the elastic moduli with only 14 modes. Another advantage of the proposed approach with the mode selection is that the convergence histories of the parameters are smoother than those without the initial condition updates and the mode selection. Thus, we can almost be sure that the converged solutions we obtain from the minimizations can be improved by adding more modes.

6. Application to laminated electrical steel sheets

In this section, we discuss the application of the proposed method to the identification of the elastic moduli of the laminated electrical steel sheets. The photograph of the test specimen is shown in Fig. 4. The test specimen consists of electrical steel sheets with nominal thickness of 0.35mm, which are bonded together by adhesive layers, where $L_1 = 0.199$m, $L_2 = 0.199$m, and $L_3 = 0.0518$m. The total number of the sheets is 146, and the total mass of the test specimen is 15.8kg. Using the specimen, modal testing was conducted. The mode shapes with the associated mode frequencies are shown in Fig. 5. For the first five modes, the mode shapes show typical out-of-plane bending vibration modes of plates. The 6th and the 7th modes, however, show slightly distorted higher-order vibration shapes that do not typically appear in low frequency ranges of the vibration modes of plates. This appears to be caused by the inhomogeneity of the microscopic adhesive layers between the electrical steel sheets. The 8th through the 12th modes are the typical in-plane bending vibration modes of plates.

Using the modes obtained by the experimental modal analysis, the proposed elastic moduli identification method was applied, and the elastic moduli of the test specimen were determined. The initial values of the elastic moduli were E_1=180GPa, E_2=180GPa, E_3=100GPa, G_{12}=76.9GPa, G_{23}=7.69GPa, G_{13}=7.69GPa, ν_{12}=0.3, ν_{23}=0.3, and ν_{13}=0.3. It is noted that preliminary studies revealed that starting the algorithm with the initial values that were used in the numerical validation would result in very slow convergence. Therefore, the initial values were chosen such that the model is "softer" in the direction of lamination (x_3), i.e., $E_3 < \{E_1, E_2\}$ and $\{G_{23}, G_{13}\} < G_{12}$. The Poisson's ratios were simply set to the typical values used for isotropic steel.

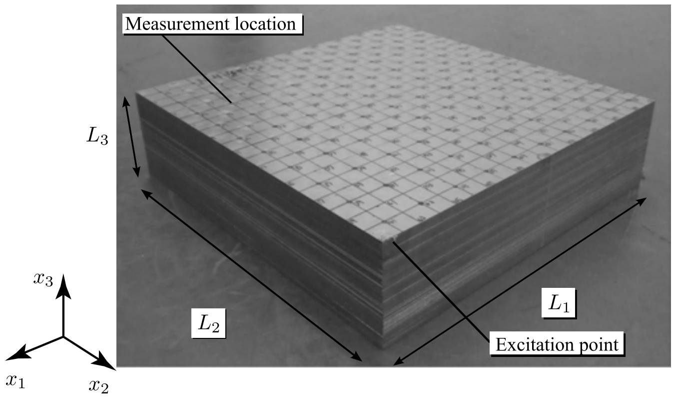
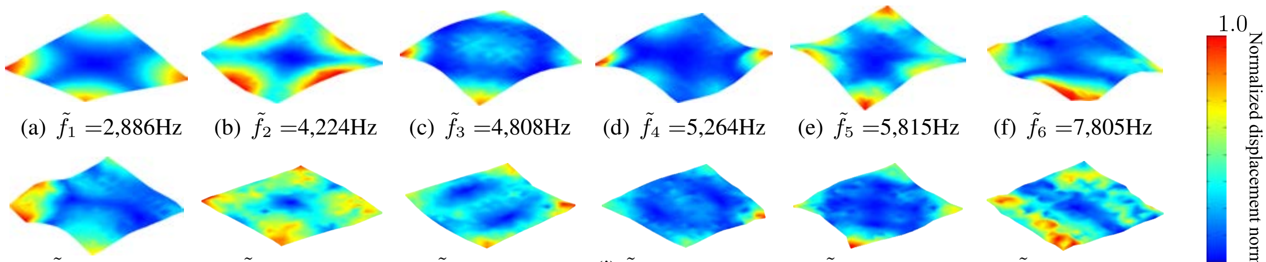


Figure 4: Photograph of the test specimen: laminated electrical steel sheets



(a) $\tilde{f}_1$ =2,886Hz (b) $\tilde{f}_2$ =4,224Hz (c) $\tilde{f}_3$ =4,808Hz (d) $\tilde{f}_4$ =5,264Hz (e) $\tilde{f}_5$ =5,815Hz (f) $\tilde{f}_6$ =7,805Hz

(g) $\tilde{f}_7$ =8,041Hz (h) $\tilde{f}_8$ =10,368Hz (i) $\tilde{f}_9$ =10,443Hz (j) $\tilde{f}_{10}$ =10,698Hz (k) $\tilde{f}_{11}$ =12,603Hz (l) $\tilde{f}_{12}$ =14,691Hz

Figure 5: Measured mode shapes and associated modal frequencies

The minimization histories of the elastic moduli with respect to the number of modes are shown in Fig. 6. The converged elastic moduli with $n = 12$ were E_1=204GPa, E_2=198GPa, E_3=99.8GPa, G_{12}=81.2GPa, G_{23}=11.7GPa, G_{13}=13.6GPa, ν_{12}=0.30, ν_{23}=0.29, and ν_{13}=0.31. The errors between the measured frequencies and those computed with the initial values $\boldsymbol{p}_0$, and the converged solution with the proposed method are shown in Fig. 7. As we can see in Fig. 7, all frequency errors decrease when the converged solution is used. Indeed, the average error computed with the initial value $\boldsymbol{p}_0$ is 11.75%, whereas that computed with the converged solution by the proposed method decreases down to 1.72%. This shows the validity of the proposed method.

Conclusions

In this paper, we proposed an elastic moduli identification method for orthotropic media based on nonlinear LS fit between the measured and the computed modal frequencies, with accelerated convergence and improved accuracy. In Section 4, the nonlinear LS problem was formulated, and a novel analysis procedure has been proposed, which

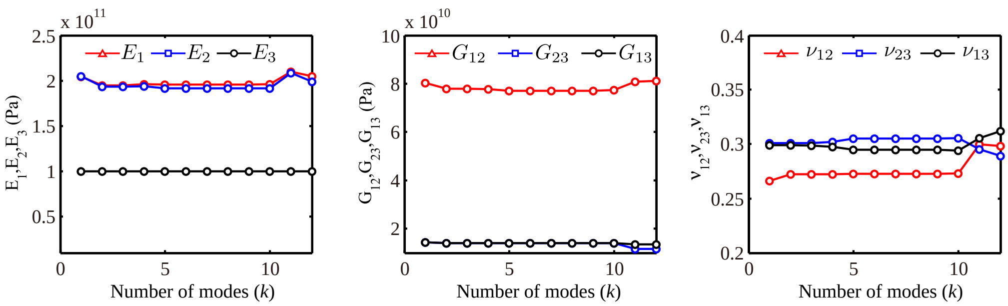


Figure 6: Convergence histories of the parameters for laminated electrical steel sheets

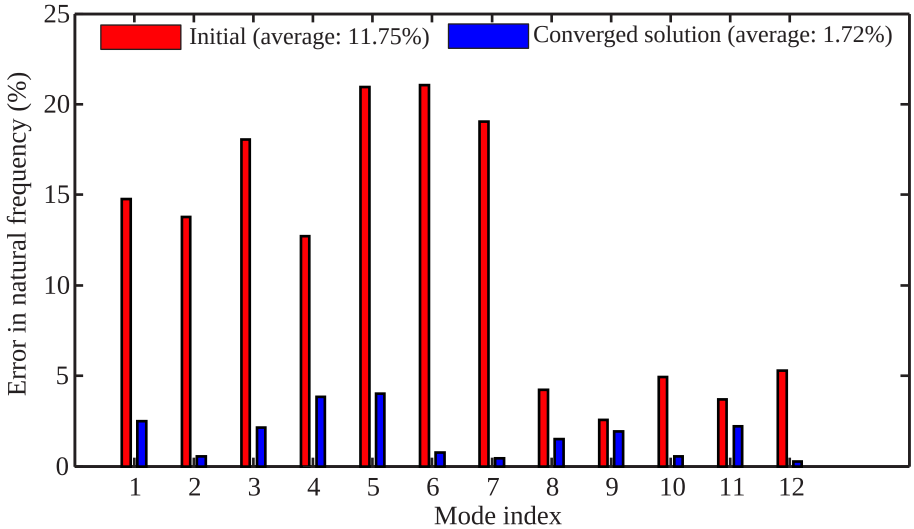
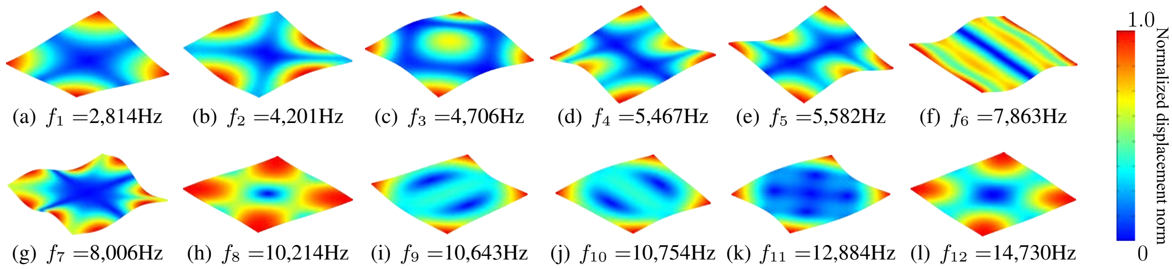


Figure 7: Errors in the predicted natural frequencies



(a) f_1 =2,814Hz (b) f_2 =4,201Hz (c) f_3 =4,706Hz (d) f_4 =5,467Hz (e) f_5 =5,582Hz (f) f_6 =7,863Hz

(g) f_7 =8,006Hz (h) f_8 =10,214Hz (i) f_9 =10,643Hz (j) f_{10} =10,754Hz (k) f_{11} =12,884Hz (l) f_{12} =14,730Hz

Figure 8: Computed mode shapes and associated modal frequencies

improves the LS solutions by successive increments of the number of modal frequencies and initial condition updates. In Section 5, the minimization method was applied to a numerical example, and its validity was confirmed. In Section 6, the method was applied to the identification of the elastic moduli of the laminated electrical steel sheets, and it successfully determined the elastic moduli where the average frequency error is minimized.

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