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Interval buckling analysis of steel structures using mathematical programming approach

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Abstract
A novel mathematical programming approach is proposed in this study to assess the linear buckling load of steel structure with uncertain system parameters. The considered uncertainties of system parameters are modelled by the interval approach such that only bounds of uncertain parameters are available. This particular uncertainty model is applicable for situations where probabilistic approach is inapplicable due to the insufficiency of the data of system parameters. By implementing an alternative finite element formulation for the two-dimensional beam element, the deterministic second order geometrically nonlinear problem is formulated into a mathematical programming problem. Furthermore, by treating all the interval uncertain system parameters as bounded mathematical programming variables, the integration of interval uncertainties in the deterministic linear buckling analysis becomes possible, such that the lower and upper bounds of the buckling load can be adequately obtained by solving two explicit nonlinear programs. The proposed computational scheme offers a single-phase interval buckling analysis for steel structures by combining the linear analysis of the structure at its reference configuration with the eigenvalue calculation. Such ability can well maintain the physical feasibility of the engineering structures for the purpose of uncertainty analysis, so the physically meaningful lower and upper bounds of the buckling load can be efficiently obtained. In addition, unlike traditional uncertain buckling analysis, the proposed method is able to thoroughly model the dependency between uncertain system parameters (i.e., the physical relationship between cross-sectional area and second moment of area of beam element must be compatible when cross-sectional area possesses uncertainty). One numerical example is presented to illustrate the accuracy and applicability of the proposed approach.
Keywords: interval analysis, buckling, steel structure, mathematical programming, dependency.

1. Introduction

Linear buckling analysis provides a computational framework which has been prevalently implemented for assessing the safety of engineering structures against large deformation. Due to its extensive applicability, computational efficiency and remarkable accuracy, linear buckling analysis has been extensively performed in modern engineering applications by integrating such analysis framework into front-edge engineering analysis software.

However, one practical issue often encountered among engineering application is the impact of uncertainties of system parameters. The existence of uncertainties of system parameters is inherent, and the impact upon the structural response is mercurial yet inevitable [1]. Such implications can influence structural performance [2], and consequently structural safety would be compromised if the impacts of uncertainties are not addressed appropriately [3].

In order to rigorously assess structural safety against large deformation, buckling analyses with considerations of uncertainties of system parameters have been proposed. Numbers of research works on the linear buckling analysis with stochastic uncertainties have been developed. However, types of uncertainties of system parameters are not unique. Such diversity of uncertainty stimulates further development of other forms of non-deterministic linear buckling analysis for various engineering situations.

This paper presents a mathematical programming based uncertain linear buckling analysis for assessing the buckling load of engineering frames which involve interval uncertain parameters. The presented method offers the worst and best case buckling loads of frames including both uncertain-but-bounded material properties and loading conditions in two explicit calculations. Uncertain linear buckling analysis is transformed into an eigenvalue problem with interval parameters within finite element (FE) framework. Furthermore, the proposed method is able to reformulate the interval eigenvalue problem into two explicit nonlinear mathematical programs (NLP), which individually depicts the feasible regions for the worst and best case buckling load. The applicability, accuracy, as well as the computational efficiency of the presented approach are illustrated through a practically motivated numerical example.

2. Deterministic linear buckling analysis

From traditional finite element method (FEM), by assuming constant axial force, the deterministic linear buckling analysis can be formulated into an eigenvalue problem as:

$$(\mathbf{K_M} + \lambda_b \mathbf{K_G})\mathbf{z} = \mathbf{0} \tag{1}$$

where $\mathbf{K_M}, \mathbf{K_G} \in \Re^{d\times d}$ denote the conventional material and geometric stiffness matrices at reference configuration respectively; d denotes the total degree of freedom of the structure; λ_b denotes the structural buckling load which is the minimum positive eigenvalue of Eq.(1); $\mathbf{z} \in \Re^d$ is a non-zero vector which denotes the eigenvector corresponding to the buckling load or the eigenvalue. Since the eigenvalue analysis defined in Eq.(1) is indeterminate, the eigenvector z denotes the shape of the buckling of engineering structure instead of actual buckled deformation [4].

In this study, an alternative FE formulation of the 2-dimensional beam is adopted. For *i*th element, the adopted FE model is illustrated in Figure 1.

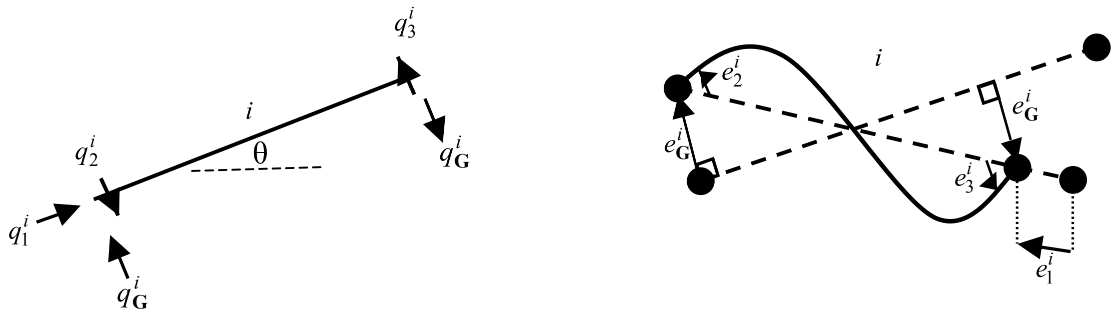

Figure 1: Generic 2D frame element *i* with second-order geometric nonlinearity (a) generalized stresses, (b) generalized strains

The adopted approach is based on the second-order geometric theory which assumes that displacements from undeformed configuration are geometrically small [5]. For a generic 2D frame element, there are four generalized stress/strain components involved in the second-order geometrically nonlinear 2D frame element. The axial and two end rotational components are adopted from linear analysis such that:

$$\mathbf{q}^i = [q_1^i \quad q_2^i \quad q_3^i]^T \in \Re^3 \tag{2}$$

$$\mathbf{e}^i = [e_1^i \quad e_2^i \quad e_3^i]^T \in \Re^3 \tag{3}$$

whereas the additional transverse component is employed for the purpose of the second-order geometrically nonlinear analysis [5], which takes the form of:

$$\mathbf{q}_\mathbf{G}^i = [q_\mathbf{G}^i] \in \Re \tag{4}$$

$$\mathbf{e}_\mathbf{G}^i = [e_\mathbf{G}^i] \in \Re \tag{5}$$

where, $\mathbf{q}^i$ and $\mathbf{q}_\mathbf{G}^i$ denote the generalized stresses which are illustrated in Figure 1(a); $\mathbf{e}^i$ and $\mathbf{e}_\mathbf{G}^i$ denote the generalized strain which are illustrated in Figure 1(b). Therefore, the equilibrium condition of the *i*th 2D frame element for the second-order geometrically nonlinear analysis is:

$$\begin{bmatrix} \cos\theta & -\sin\theta/L_i & -\sin\theta/L_i \\ \sin\theta & \cos\theta/L_i & \cos\theta/L_i \\ 0 & 1 & 0 \\ -\cos\theta & \sin\theta/L_i & \sin\theta/L_i \\ -\sin\theta & -\cos\theta/L_i & -\cos\theta/L_i \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1^i \\ q_2^i \\ q_3^i \end{bmatrix} + \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} q_\mathbf{G}^i = \begin{bmatrix} F_1^i \\ F_2^i \\ F_3^i \\ F_4^i \\ F_5^i \\ F_6^i \end{bmatrix} \tag{6}$$

or

$$\mathbf{C}_0^{iT}\mathbf{q}^i + \mathbf{C}_\mathbf{G}^{iT}\mathbf{q}_\mathbf{G}^i = \mathbf{F}^i \tag{7}$$

where L_i is the length of the ith element. The elemental compatibility condition is defined as:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 & -\cos\theta & -\sin\theta & 0 \\ -\sin\theta/L^i & \cos\theta/L^i & 1 & \sin\theta/L^i & -\cos\theta/L^i & 0 \\ -\sin\theta/L^i & \cos\theta/L^i & 0 & \sin\theta/L^i & -\cos\theta/L^i & 1 \end{bmatrix} \begin{bmatrix} u_1^i \\ u_2^i \\ u_3^i \\ u_4^i \\ u_5^i \\ u_6^i \end{bmatrix} = \begin{bmatrix} e_1^i \\ e_2^i \\ e_3^i \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} -\sin\theta & \cos\theta & 0 & \sin\theta & -\cos\theta & 0 \end{bmatrix} \begin{bmatrix} u_1^i \\ u_2^i \\ u_3^i \\ u_4^i \\ u_5^i \\ u_6^i \end{bmatrix} = e_\mathbf{G}^i \tag{9}$$

or

$$\mathbf{C}_0^i\mathbf{u}^i = \mathbf{e}^i \tag{10}$$

$$\mathbf{C}_\mathbf{G}^i\mathbf{u}^i = \mathbf{e}_\mathbf{G}^i \tag{11}$$

and the constitutive condition is defined as:

$$\left\{ \begin{bmatrix} E_iA_i/L_i & 0 & 0 \\ 0 & 4E_iI_i/L_i & 2E_iI_i/L_i \\ 0 & 2E_iI_i/L_i & 4E_iI_i/L_i \end{bmatrix} + (-q_1^i) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2L_i/15 & -L_i/30 \\ 0 & -L_i/30 & 2L_i/15 \end{bmatrix} \right\} \begin{bmatrix} e_1^i \\ e_2^i \\ e_3^i \end{bmatrix} = \begin{bmatrix} q_1^i \\ q_2^i \\ q_3^i \end{bmatrix} \tag{12}$$

$$\left(\frac{-q_1^i}{L_i} \right) e_\mathbf{G}^i = q_\mathbf{G}^i \tag{13}$$

or

$$(\mathbf{S}_0^i + \mathbf{S}_\mathbf{G}^i)\mathbf{e}^i = \mathbf{q}^i \tag{14}$$

$$\mathbf{S}_\mathbf{F}^i\mathbf{e}_\mathbf{G}^i = \mathbf{q}_\mathbf{G}^i \tag{15}$$

where E_i, A_i, I_i are the Young's modulus, cross-sectional area and the second moment of area of the ith element, respectively. Eqs.(6)-(15) alternatively formulate the three governing equations for the second-order geometrically nonlinear 2D frame element. This unconventional formulation is equivalent to the governing equation formulated by the traditional FEM. For example, let $\theta = 0$ and substitute Eqs.(10), (11) (14) and (15) into Eq.(7), thus

$$\begin{aligned} \mathbf{F}^i &= \mathbf{C}_0^{iT}\mathbf{q}^i + \mathbf{C}_\mathbf{G}^{iT}\mathbf{q}_\mathbf{G}^i \\ &= \mathbf{C}_0^{iT}(\mathbf{S}_0^i + \mathbf{S}_\mathbf{G}^i)\mathbf{e}^i + \mathbf{C}_\mathbf{G}^{iT}\mathbf{S}_\mathbf{F}^i\mathbf{e}_\mathbf{G}^i \\ &= \mathbf{C}_0^{iT}(\mathbf{S}_0^i + \mathbf{S}_\mathbf{G}^i)\mathbf{C}_0^i\mathbf{u}^i + \mathbf{C}_\mathbf{G}^{iT}\mathbf{S}_\mathbf{F}^i\mathbf{C}_\mathbf{G}^i\mathbf{u}^i \\ &= \mathbf{C}_0^{iT}\mathbf{S}_0^i\mathbf{C}_0^i\mathbf{u}^i + (\mathbf{C}_0^{iT}\mathbf{S}_\mathbf{G}^i\mathbf{C}_0^i + \mathbf{C}_\mathbf{G}^{iT}\mathbf{S}_\mathbf{F}^i\mathbf{C}_\mathbf{G}^i)\mathbf{u}^i \\ &= (\mathbf{K}_\mathbf{M}^i + \mathbf{K}_\mathbf{G}^i)\mathbf{u}^i \end{aligned} \tag{16}$$

where $\mathbf{K}_\mathbf{M}^i = \mathbf{C}_0^{iT}\mathbf{S}_0^i\mathbf{C}_0^i$, and $\mathbf{K}_\mathbf{G}^i = \mathbf{C}_0^{iT}\mathbf{S}_\mathbf{G}^i\mathbf{C}_0^i + \mathbf{C}_\mathbf{G}^{iT}\mathbf{S}_\mathbf{F}^i\mathbf{C}_\mathbf{G}^i$. Eq.(16) coincides with the traditional FE formulation for linear buckling analysis. Therefore, the eigenvalue problem of the linear buckling analysis of 2D frame with n elements can be alternatively expressed as:

$$(\mathbf{K_M} + \lambda_{\mathrm{cr}}\mathbf{K_G})\mathbf{z} = \mathbf{C}_0^T\mathbf{S}_0\mathbf{C}_0\mathbf{z} + \lambda_{\mathrm{cr}}(\mathbf{C}_0^T\mathbf{S_G}\mathbf{C}_0 + \mathbf{C_G}^T\mathbf{S_F}\mathbf{C_G})\mathbf{z} = 0 \tag{17}$$

where for a second-order geometrically nonlinear 2D frame, $\mathbf{C}_0 \in \Re^{3n\times d}$ and $\mathbf{C_G} \in \Re^{n\times d}$ are the two global compatibility matrices, and their transposes are the global equilibrium matrices; $\mathbf{S}_0, \mathbf{S_G} \in \Re^{3n\times 3n}$ and $\mathbf{S_F} \in \Re^{n\times n}$ are the global deterministic stiffness matrices calculated at reference configuration for 2D frames. Eq.(17) presents the alternative formulation for the deterministic linear buckling analysis which is beneficial for interval linear buckling analysis.

3. Solution algorithm of uncertain linear buckling analysis of frame structure

The uncertain parameters considered in this investigation are including the Young's modulus, cross-sectional area and second moment of inertia of each structural element, as well as the externally applied loadings at reference configuration.

Therefore, by considering uncertainties of parameters, the worst case buckling load can be calculated by solving:

$$\min\ \lambda_{\mathrm{b}}$$

$$\text{subject to:}$$

$$\begin{cases} \mathbf{C}_0^T\mathbf{S}_0(\mathbf{E},\mathbf{A},\mathbf{I})\mathbf{C}_0\mathbf{z} + \lambda_{\mathrm{b}}[\mathbf{C}_0^T\mathbf{S_G}(\mathbf{F_{ref}})\mathbf{C}_0 + \mathbf{C_G}^T\mathbf{S_F}(\mathbf{F_{ref}})\mathbf{C_G}]\mathbf{z} = 0 \\ \mathbf{E} \in \tilde{\mathbf{E}} := \{\mathbf{E} \in \Re^n \mid \underline{E_i} \le E_i \le \overline{E_i}, \text{for } i = 1...n\} \\ \mathbf{A} \in \tilde{\mathbf{A}} := \{\mathbf{A} \in \Re^n \mid \underline{A_i} \le A_i \le \overline{A_i}, \text{for } i = 1...n\} \\ \mathbf{F_{ref}} \in \tilde{\mathbf{F}} := \{\mathbf{F_{ref}} \in \Re^d \mid \underline{F}_{\mathrm{ref}}^j \le F_{\mathrm{ref}}^j \le \overline{F}_{\mathrm{ref}}^j, \text{for } j = 1...d\} \\ \mathbf{I} = f(\mathbf{A}) \\ \|\mathbf{z}\|_2 = 1 \end{cases} \tag{18}$$

and the best case solution can be determined as:

$$\max\ \lambda_{\mathrm{b}}$$

$$\text{subject to:}$$

$$\begin{cases} \mathbf{C}_0^T\mathbf{S}_0(\mathbf{E},\mathbf{A},\mathbf{I})\mathbf{C}_0\mathbf{z} + \lambda_{\mathrm{b}}[\mathbf{C}_0^T\mathbf{S_G}(\mathbf{F_{ref}})\mathbf{C}_0 + \mathbf{C_G}^T\mathbf{S_F}(\mathbf{F_{ref}})\mathbf{C_G}]\mathbf{z} = 0 \\ \mathbf{E} \in \tilde{\mathbf{E}} := \{\mathbf{E} \in \Re^n \mid \underline{E_i} \le E_i \le \overline{E_i}, \text{for } i = 1...n\} \\ \mathbf{A} \in \tilde{\mathbf{A}} := \{\mathbf{A} \in \Re^n \mid \underline{A_i} \le A_i \le \overline{A_i}, \text{for } i = 1...n\} \\ \mathbf{F_{ref}} \in \tilde{\mathbf{F}} := \{\mathbf{F_{ref}} \in \Re^d \mid \underline{F}_{\mathrm{ref}}^j \le F_{\mathrm{ref}}^j \le \overline{F}_{\mathrm{ref}}^j, \text{for } j = 1...d\} \\ \mathbf{I} = f(\mathbf{A}) \\ \|\mathbf{z}\|_2 = 1 \\ z^i \cdot z_{\mathrm{b,det}}^i \ge 0 \ \textit{for } i = 1 \ldots d \end{cases} \tag{19}$$

where $z_{\mathrm{b,det}}^i$ denotes the ith component of the deterministic buckling mode vector, which is the eigenvector corresponding to $\lambda_{\mathrm{b,det}}$. In addition, the constraint $\mathbf{I} = f(\mathbf{A})$ is introduced to model the dependency between the cross-sectional area and the second moment of area of the same structural element.

Eqs.(18) and (19) provide a mathematical programming based approach for calculating the worst and best case buckling loads for 2D frames. The proposed method transforms the uncertain linear buckling analysis, which in essence is an interval eigenvalue problem, into two explicit NLPs. By adopting the formulation of the generalized stress/strain with the unique structural characteristics of the stiffness matrices, the interval parameters are able to be extracted out from the stiffness matrices and explicitly modelled as mathematical programming variables. Unlike traditional interval analysis, the proposed method involves no interval arithmetic such that the sharpness of the worst and best buckling loads are not compromised due to the inveterate issue of dependency associated with interval arithmetic [6].

4. Numerical example

In order to illustrate the applicability, accuracy, and efficiency of the presented computational approach, one practically motivated frame structure is investigated. The reference configuration of the structure is shown in Figure 2. The NLPs involved in both worst case and best case calculations are solved by a commercial NLP solver named CONOPT [7], which implemented within a sophisticated modelling environment named the general algebraic modelling system or GAMS [8].

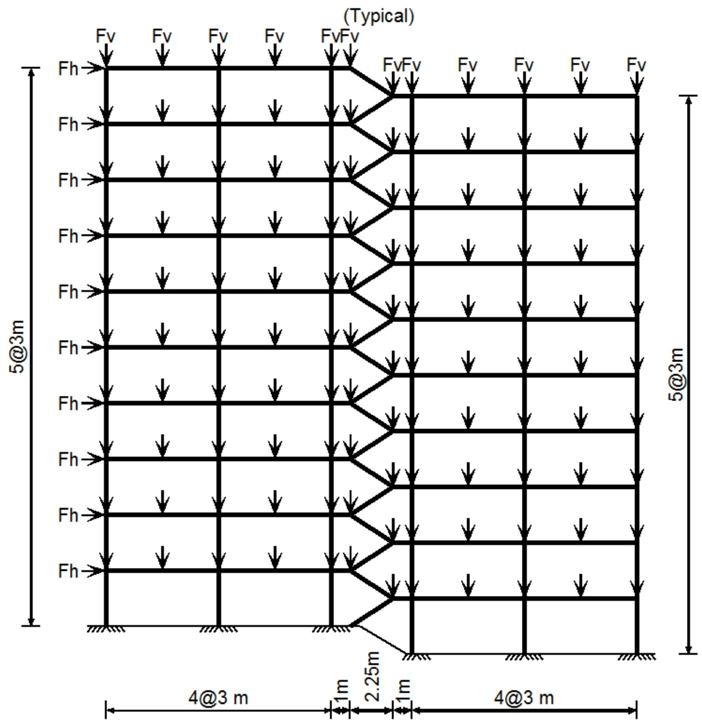

Figure 2: Ten-storey five-bay frame

The considered uncertain parameters are including the Young's modulus, cross-sectional area of beam and column, second moment of inertia of beam and column, as well as the applied loadings at reference configuration. All the information on the uncertain parameters has been presented in Table 1.

Table 1: Interval parameters of ten-storey five-bay frame

Interval Parameters	Lower bound	Upper bound
E	176 GPa	224 GPa
$\mathbf{A_c}$	$326.8\times10^{-4}\,m^2$	$361.2\times10^{-4}\,m^2$
$\mathbf{A_b}$	$56.34\times10^{-4}\,m^2$	$62.26\times10^{-4}\,m^2$
$\mathbf{F_v}$	68kN	92kN
$\mathbf{F_h}$	24kN	36kN

For the ten-storey frame shown in Figure 1, 400WC270 has been implemented to model all columns whereas beams are modelled by 310UB46.2 [9]. In order to maintain the physical feasibility between the cross-sectional area and second moment of inertia for the same element, the following compatibility conditions are introduced as:

$$I_c(A_c) = 0.1898A_c^2 + 0.0241A_c - 2\times10^{-5} \tag{20}$$

for all the 400WC270 columns, and

$$I_b(A_b) = -0.8876A_b^2 + 0.0288A_b - 4\times10^{-5} \tag{21}$$

for all the 310UB46.2 beams.

The worst and best structural buckling load calculated by the proposed NLP approach are $\lambda_{b,worst}^{NLP} = 27.81$ and $\lambda_{b,best}^{NLP} = 53.34$ respectively. Due to the unavailability of analytical solution for complex structure such as the one in current example, the Monte-Carlo simulation method with 100,000 simulations has been performed to partially verify the accuracy of the proposed method. The results reported by 100,000 simulations are $\lambda_{b,worst}^{mcs} = 36.01$ for the worst case, and $\lambda_{b,best}^{mcs} = 40.55$ for the best case. It is obvious that the performance of the Monte-Carlo simulation with 100,000 iterations provides enclosed solutions and the computational efficiency consumed is far more than the proposed NLP method.

5. Conclusion

Uncertain linear buckling analysis with interval parameters has been investigated. A mathematical programming founded approach is presented to assess the buckling loads of engineering frames against undesirably large displacement by calculating the worst and best case buckling loads.

All interval parameters considered in this study are able to be modelled as mathematical programming variables with upper and lower bounds through reformulations of the traditional FE approach. The advantage is that the interval dependence associated with interval arithmetic can be completely eliminated, so the sharpness of the extremity of the buckling loads can be enhanced.

For situations, such as eigen-buckling analyses of structures involving repeated eigenvalues, structures with closely spaced eigenvalues, as well as defective structural systems etc, have not been investigated in the present study.

6. Acknowledgements

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6. References

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