Large-scale robust topology optimization under load-uncertainty

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1. Abstract

Structures designed by topology optimization (TO) are frequently sensitive to loads different from the ones accounted for in the optimization. In extreme cases this means that loads differing ever so slightly from the ones it was designed to carry may cause a structure to collapse. It is therefore clear that handling uncertainty regarding the actual loadings is important. To address this issue in a systematic manner is one of the main goals in the field of robust TO. In this work we present a deterministic robust formulation of TO for maximum stiffness design which accounts for uncertain variations around a set of nominal loads. The idea is to find a design which minimizes the maximum compliance obtained as the loads vary in infinite, so-called uncertainty sets. This naturally gives rise to a semi-infinite optimization problem, which we here reformulate into a non-linear, semi-definite program. With appropriate numerical algorithms this optimization problem can be solved at a cost similar to that of solving a standard multiple load-case TO problem with the number of loads equal to the number of spatial dimensions plus one, times the number of nominal loads. In contrast to most previously suggested methods, which can only be applied to small-scale problems, the presented method is – as illustrated by a numerical example – well-suited for large-scale TO problems.

2. Keywords: Robust optimization, Topology optimization, Large-scale optimization, Non-linear semi-definite programming

3. Introduction

Robust problem formulations in structural optimization can be divided into two groups [3]: (i) *stochastic* and (ii) *deterministic*, or worst-case. The stochastic approach is perhaps the most common. It can essentially be divided into two classes: reliability-based methods [15], often based on an inner loop where the structure is optimized and an outer loop where the uncertainty parameters are modified, and ”sampling-based” methods, where the stochastic problem is converted into a deterministic by sampling from a probability distribution [10]. A drawback of stochastic methods is their reliance on good statistical data, and, in certain applications more importantly [3], they only provide probabilistic guarantees of robustness.

In this paper we consider robust deterministic topology optimization of discretized continuum bodies governed by a linear equilibrium equation of the form

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}_0 + \boldsymbol{f}_{var}$$

where $\boldsymbol{f}_0$ is a given fixed load and $\boldsymbol{f}_{var}$ is allowed to vary in some uncertainty set. The special case when $\boldsymbol{f}_0 = \boldsymbol{0}$ has been treated in both finite- and infinite-dimensional settings by several authors [2, 8, 17, 5, 13]. The general case with a non-zero $\boldsymbol{f}_0$ is however less explored. Ben-Tal et al. [3, p. 215] showed how to formulate the problem as a semi-definite program, which was however not suitable for large-scale problems (see also [7]). Kanno [14] proposed an optimization problem in the form of a mathematical program with equilibrium constraints (MPEC). That formulation resembles the so-called simultaneous formulation in traditional stiffness optimization [9] involving the equilibrium equation as an explicit constraint; consequently it may be difficult to solve large-scale instances of the MPEC-formulation. To the authors' knowledge, the only paper dealing with large-scale, deterministic robust structural optimization is [12]. Therein the authors propose a special algorithm to solve a generalized, inhomogeneous eigenvalue problem arising from the maximization of the compliance. That algorithm however requires up to 15 factorizations of a system matrix (involving the stiffness matrix) for each design, while the method we propose requires just one. In addition, the method proposed herein is readily implemented using standard linear solvers and software for non-linear programming.

In the following the space of symmetric $n \times n$ matrices with real entries is denoted $\mathbb{S}^n$. For $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathbb{S}^n$, ”$\boldsymbol{A} \succeq \boldsymbol{B}$” (”$\boldsymbol{A} \succ \boldsymbol{B}$”) means that $\boldsymbol{A} - \boldsymbol{B}$ is positive semi-definite (positive definite). The set of symmetric, positive semi-definite matrices is denoted $\mathbb{S}^n_+$. The Euclidean vector norm is denoted $||\cdot||$, $\mathbb{R}_+$ is the set of non-negative real numbers, and $\boldsymbol{I}$ denotes an identity matrix of appropriate size.

4. The model

We consider a linearly elastic continuum body divided into m elements. Associated with each element is a design variable x_i which determines whether there is material in the element or not. The design variables are collected in a vector $\boldsymbol{x}$ of length m. To avoid numerical issues such as checkerboards and mesh dependency a different set of variables, referred to as physical variables, are used to compute the stiffness matrix (and mass). These variables are related to the design variables through a so-called filter [6], using which the i-th physical variable is written as

$$\rho_i(\boldsymbol{x}) = \sum_{j=1}^{m} \Psi_{ij} x_j, \quad \Psi_{ij} = \frac{\phi_{ij} v_j}{\sum_{k=1}^{m} \phi_{ik} v_k}, \quad \phi_{ij} = \max\left(0, R - ||\boldsymbol{e}_i - \boldsymbol{e}_j||\right),$$

where v_j and $\boldsymbol{e}_j$ denotes the volume and the position of the geometric center, respectively, of element j, and R is the filter radius.

The structure is assumed to be in quasi-static equilibrium undergoing small deformations when subject to a nodal load vector $\boldsymbol{f}_r$. By the standard displacement-based FEM the nodal displacement vector $\boldsymbol{u}$ should therefore satisfy

$$\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}_r, \tag{1}$$

where, following the SIMP-approach to TO, the stiffness matrix $\boldsymbol{K}(\boldsymbol{x}) = \sum_{i=1}^{m} \rho_i(\boldsymbol{x})^p \boldsymbol{K}_i$, in which p is a positive constant and $\boldsymbol{K}_i \in \mathbb{S}^n_+$ are element stiffness matrices. Here n denotes the number of degrees of freedom. We assume that rigid body motions are prevented, so $\boldsymbol{K}(\boldsymbol{1}) \succ \boldsymbol{0}$, where $\boldsymbol{1}$ is a vector of ones.

The load vector in (1) is defined as

$$\boldsymbol{f}_r = \boldsymbol{f}_0 + \boldsymbol{B}^\mathsf{T}\boldsymbol{r}, \tag{2}$$

where $\boldsymbol{r} \in T = \{\boldsymbol{r} \in \mathbb{R}^d \,|\, ||\boldsymbol{r}|| \leq 1\}$, with d the number of spatial dimensions. The vector $\boldsymbol{f}_0$, the nominal load, is fixed during optimization, while $\boldsymbol{B}^\mathsf{T}\boldsymbol{r}$, where $\boldsymbol{B}^\mathsf{T} \in \mathbb{R}^{n\times d}$, is allowed to vary — it is through this variation that uncertainty is accounted for. The (fixed) matrix $\boldsymbol{B}$ can be used to specify which nodes are subject to load-uncertainty, and in which direction the uncertainty (if any) is greatest.

5. Optimization problem

We begin by defining the worst-case compliance as

$$\max_{\boldsymbol{r}\in T} \boldsymbol{f}_r^\mathsf{T} \boldsymbol{u}_r(\boldsymbol{x}), \tag{3}$$

where $\boldsymbol{u}_r(\boldsymbol{x})$ solves (1). Assuming constant density, the problem of minimizing the worst-case compliance with an upper bound on the weight can now be written as

$$\min_{\boldsymbol{x}\in\mathcal{H}} \max_{||\boldsymbol{r}||\leq 1} \boldsymbol{f}_r^\mathsf{T} \boldsymbol{K}(\boldsymbol{x})^{-1} \boldsymbol{f}_r,$$

where $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^m \,|\, \varepsilon \leq x_i \leq 1, i = 1,\ldots,m,\ \sum_{i=1}^{m} v_i\rho_i(\boldsymbol{x}) \leq v\}$, in which the positive constant v is the upper bound on the volume. The small positive constant ε ensures that the stiffness matrix is positive definite if the structure is appropriately supported.

A natural thing to do when confronted with a min-max problem is to reformulate it into a bound formulation:

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathcal{H}, z\in\mathbb{R}_+} && z \\ &\text{subject to} && \boldsymbol{f}_r^\mathsf{T} \boldsymbol{K}(\boldsymbol{x})^{-1} \boldsymbol{f}_r \leq z, \quad \forall \boldsymbol{r} \in \mathbb{R}^d : ||\boldsymbol{r}|| \leq 1, \end{aligned} \tag{4}$$

where z is an additional variable and ":" should be read as "such that". Since there is now an infinite number of constraints it is not immediately clear that anything has been gained from this reformulation. However, the semi-infinite problem (4) can be reformulated as a non-linear semi-definite program using the following result.

Theorem 1. $\boldsymbol{f}_r^\mathsf{T} \boldsymbol{K}(\boldsymbol{x})^{-1} \boldsymbol{f}_r \leq z$ for all $\boldsymbol{r} \in \mathbb{R}^d : ||\boldsymbol{r}|| \leq 1$ if and only if there exists a $\lambda \in \mathbb{R}_+$ such that

$$\begin{pmatrix} \lambda \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & z-\lambda \end{pmatrix} - \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{f}_0^\mathsf{T} \end{pmatrix} \boldsymbol{K}(\boldsymbol{x})^{-1} \begin{pmatrix} \boldsymbol{B}^\mathsf{T} & \boldsymbol{f}_0 \end{pmatrix} \succeq \boldsymbol{0}.$$

To prove Theorem 1 we use two lemmas:

Lemma 1. [S-lemma] (See [3, p. 483] for a proof.) For $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ in $\mathbb{S}^n$, with $\boldsymbol{y}_0^\mathsf{T}\boldsymbol{A}_1\boldsymbol{y}_0 > 0$ for some $\boldsymbol{y}_0$,

$$\boldsymbol{y}^\mathsf{T}\boldsymbol{A}_1\boldsymbol{y} \geq 0 \Rightarrow \boldsymbol{y}^\mathsf{T}\boldsymbol{A}_2\boldsymbol{y} \geq 0 \;\Leftrightarrow\; \exists\lambda \in \mathbb{R}_+ \;:\; \boldsymbol{A}_2 \succeq \lambda\boldsymbol{A}_1.$$

Lemma 2. For $\boldsymbol{A} \in \mathbb{R}^{n\times n}$,

$$\begin{pmatrix} \boldsymbol{y} \\ 1 \end{pmatrix}^\mathsf{T} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^\mathsf{T} & c \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ 1 \end{pmatrix} \geq 0 \quad \forall \boldsymbol{y} : ||\boldsymbol{y}||^2 \leq 1 \;\Leftrightarrow\; \begin{pmatrix} \boldsymbol{y} \\ w \end{pmatrix}^\mathsf{T} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^\mathsf{T} & c \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ w \end{pmatrix} \geq 0 \quad \forall (\boldsymbol{y}, w) : ||\boldsymbol{y}||^2 \leq w^2.$$

Proof. The implication from right to left follows immediately by taking $w = 1$. Now consider going from left to right, beginning with

$$\begin{pmatrix} \boldsymbol{y} \\ 1 \end{pmatrix}^\mathsf{T} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^\mathsf{T} & c \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ 1 \end{pmatrix} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A}\boldsymbol{y} + 2\boldsymbol{b}^\mathsf{T}\boldsymbol{y} + c \geq 0 \quad \forall \boldsymbol{y} : ||\boldsymbol{y}||^2 \leq 1. \tag{5}$$

Let $\boldsymbol{y} = \boldsymbol{u}/w$, with $||\boldsymbol{u}||^2 \leq w^2$ and $w \neq 0$. Clearly $||\boldsymbol{y}||^2 = ||\boldsymbol{u}/w||^2 = ||\boldsymbol{u}||^2/w^2 \leq 1$. Substituting $\boldsymbol{y} = \boldsymbol{u}/w$ in (5) gives

$$\boldsymbol{u}^\mathsf{T}\boldsymbol{A}\boldsymbol{u} + 2\boldsymbol{b}^\mathsf{T}\boldsymbol{u}w + cw^2 \geq 0, \quad \forall (\boldsymbol{u}, w) : ||\boldsymbol{u}||^2 \leq w^2, w \neq 0.$$

Going back to the statement of the lemma it is clear that $w = 0$ implies $\boldsymbol{y} = \boldsymbol{0}$, reducing the right side to the trivial inequality $0 \geq 0$. □

Proof of Theorem 1. Straightforward algebraic manipulations show that (the argument "$\boldsymbol{x}$" is omitted)

$$\begin{aligned} \boldsymbol{f}_r^\mathsf{T}\boldsymbol{K}^{-1}\boldsymbol{f}_r \leq z, &\qquad \forall \boldsymbol{r} \in \mathbb{R}^d : ||\boldsymbol{r}|| \leq 1 \;\Leftrightarrow \\ z - \boldsymbol{f}_0^\mathsf{T}\boldsymbol{K}^{-1}\boldsymbol{f}_0 - \boldsymbol{r}^\mathsf{T}\boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{B}^\mathsf{T}\boldsymbol{r} - 2\boldsymbol{r}^\mathsf{T}\boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{f}_0 \geq 0 &\qquad \forall \boldsymbol{r} \in \mathbb{R}^d : ||\boldsymbol{r}|| \leq 1. \end{aligned}$$

By Lemma 2 the last statement is equivalent to

$$w^2\big(z - \boldsymbol{f}_0^\mathsf{T}\boldsymbol{K}^{-1}\boldsymbol{f}_0\big) - \boldsymbol{y}^\mathsf{T}\boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{B}^\mathsf{T}\boldsymbol{y} - 2\boldsymbol{y}^\mathsf{T}\boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{f}_0 w \geq 0, \quad \forall (\boldsymbol{y}, w) \in \mathbb{R}^{d+1} : ||\boldsymbol{y}||^2 \leq w^2. \tag{6}$$

Since the condition $||\boldsymbol{y}||^2 \leq w^2$ can be written as

$$\begin{pmatrix} \boldsymbol{y} \\ w \end{pmatrix}^\mathsf{T} \boldsymbol{A}_1 \begin{pmatrix} \boldsymbol{y} \\ w \end{pmatrix} \geq 0, \quad \text{where } \boldsymbol{A}_1 = \begin{pmatrix} -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix},$$

Lemma 1 can be invoked to conclude that (6) is equivalent to

$$\exists\lambda \in \mathbb{R}_+ \;:\; \begin{pmatrix} \lambda\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & z - \lambda \end{pmatrix} - \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{f}_0^\mathsf{T} \end{pmatrix} \boldsymbol{K}^{-1} \Big(\boldsymbol{B}^\mathsf{T} \; \boldsymbol{f}_0\Big) \succeq \boldsymbol{0}.$$

□

Remark. A different proof of Theorem 1 can be obtained by using Theorem 8.2.3 in [3] together with the Schur-complement theorem [3, Lemma 6.3.4]. By the former theorem one can also obtain a formulation where $\boldsymbol{K}$ need not be invertible [7, Proposition 2],[3, p. 215]; the price to pay is a matrix inequality of large size, something which is, at present, difficult to handle numerically.

Using Theorem 1 we now replace (4) by the following problem:

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathcal{H},\, z\in\mathbb{R}_+,\, \lambda\in\mathbb{R}_+} \quad z \\ &\text{subject to} \begin{pmatrix} \lambda\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & z - \lambda \end{pmatrix} - \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{f}_0^\mathsf{T} \end{pmatrix} \boldsymbol{K}(\boldsymbol{x})^{-1} \Big(\boldsymbol{B}^\mathsf{T} \; \boldsymbol{f}_0\Big) \succeq \boldsymbol{0}. \end{aligned} \tag{7}$$

It can be shown [4, Prop. 5.72 (i)] that if $\boldsymbol{K}$ is linear in $\boldsymbol{x}$, the matrix inequality defines a convex set, making (7) a convex problem. An extension to include multiple loads, each in the form of a nominal load plus an uncertainty, is straightforward, but for notational simplicity we have refrained from doing this here.

Theorem 2. The set of globally optimal solutions to (7) is non-empty and compact.

Proof. Take $\boldsymbol{x}_0 \in \mathcal{H}$ and let

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{f}_0^\mathsf{T} \end{pmatrix} \boldsymbol{K}(\boldsymbol{x})^{-1} \Big(\boldsymbol{B}^\mathsf{T} \; \boldsymbol{f}_0\Big), \tag{8}$$

and $\boldsymbol{A}_0 = \boldsymbol{A}(\boldsymbol{x}_0)$. Let $\lambda_0 = \lambda_{max}(\boldsymbol{A}_0) \geq 0$ and $z_0 = 2\lambda_{max}(\boldsymbol{A}_0) \geq 0$, where $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue. The inequality

$$\boldsymbol{A}_0 \preceq \lambda_{max}(\boldsymbol{A}_0)\boldsymbol{I} = \begin{pmatrix} \lambda_0 \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & z_0 - \lambda_0 \end{pmatrix},$$

then shows that the feasible set of (7) is non-empty.

The inequality $\boldsymbol{A}(\boldsymbol{x}) \succeq \boldsymbol{0}$ holds for all $\boldsymbol{x} \in \mathcal{H}$ and implies that $z \geq \lambda$ for any feasible $(\boldsymbol{x}, z, \lambda)$. The globally optimal solutions to (7) are thus contained in the set

$$\mathcal{F} = \{(\boldsymbol{x}, z, \lambda) \mid \boldsymbol{x} \in \mathcal{H},\ 0 \leq z \leq z_0,\ 0 \leq \lambda \leq z_0,\ (\boldsymbol{x}, z, \lambda) \text{ satisfies the matrix inequality in (7)}\}.$$

We now consider the problem

$$\min_{(\boldsymbol{x},z,\lambda)\in\mathcal{F}} \quad z,$$

which has the same set of globally optimal solutions as (7). Since $\mathcal{F}$ is the intersection of a compact and a closed set, hence compact, and the objective function is lower semi-continuous, existence of a non-empty, compact set of globally optimal solutions follows from Weierstrass' theorem [1, Theorem 4.7]. □

The proof of Theorem 2 shows how a feasible initial point for (7) may be chosen, and provides upper bounds on z and λ.

Remark. The worst-case uncertainty vector, and therefore the worst-case load vector $\boldsymbol{f}_r$, depends on the design through

$$\boldsymbol{r}(\boldsymbol{x}) \in \arg\max_{||\boldsymbol{r}||\leq 1} (\boldsymbol{r}^\mathsf{T}\boldsymbol{B} + \boldsymbol{f}_0^\mathsf{T})\boldsymbol{K}(\boldsymbol{x})^{-1}(\boldsymbol{f}_0 + \boldsymbol{B}^\mathsf{T}\boldsymbol{r}) = \arg\max_{||\boldsymbol{r}||\leq 1} \left[\boldsymbol{r}^\mathsf{T}\boldsymbol{H}(\boldsymbol{x})\boldsymbol{r} + 2\boldsymbol{f}_0^\mathsf{T}\boldsymbol{K}(\boldsymbol{x})^{-1}\boldsymbol{B}^\mathsf{T}\boldsymbol{r}\right],$$

where $\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{B}\boldsymbol{K}(\boldsymbol{x})^{-1}\boldsymbol{B}^\mathsf{T}$. The first-order necessary optimality conditions for this problem read

$$\boldsymbol{H}(\boldsymbol{x})\boldsymbol{r} - \boldsymbol{b}(\boldsymbol{x}) = \mu\boldsymbol{r},\ \ ||\boldsymbol{r}|| = 1, \tag{9}$$

where $\boldsymbol{b}(\boldsymbol{x}) = -2\boldsymbol{B}\boldsymbol{K}(\boldsymbol{x})^{-1}\boldsymbol{f}_0$ and $\mu \in \mathbb{R}_+$ is a Lagrange multiplier. (9) is an inhomogeneous eigenvalue problem [16]; $\boldsymbol{r}(\boldsymbol{x})$ is an eigenvector associated with the maximum eigenvalue. Since the multiplicity of the latter may be greater than one we do not expect $\boldsymbol{x} \to \boldsymbol{r}(\boldsymbol{x})$ to be smooth in general.

6. Numerical treatment

To evaluate the left-hand side of the matrix constraint, rather than forming $\boldsymbol{K}(\boldsymbol{x})^{-1}$ explicitly, we solve the linear system

$$\boldsymbol{K}(\boldsymbol{x})\boldsymbol{U} = \begin{pmatrix} \boldsymbol{B}^\mathsf{T} & \boldsymbol{f}_0 \end{pmatrix} \tag{10}$$

for $\boldsymbol{U} \in \mathbb{R}^{n\times(d+1)}$. The computational cost, which is dominated by the factorization of $\boldsymbol{K}(\boldsymbol{x})$, for this is the same as that of solving the equilibrium equations in a standard multiple-load case formulation with $d+1$ loads.

Scaling is important when solving optimization problems numerically. To this end, in view of the proof of Theorem 2, we introduce new variables $z := z/z_0$ and $\lambda := \lambda/z0$, where $z_0 = 2\lambda_{max}(\boldsymbol{A}_0)$, with $\boldsymbol{A}_0$ defined in the proof of Theorem 2. Now (7) can be replaced by

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathcal{H},\, z\in[0,1],\, \lambda\in[0,1]} \quad z \\ &\text{subject to} \quad \begin{pmatrix} \lambda\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & z-\lambda \end{pmatrix} - \frac{1}{z_0}\begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{f}_0^\mathsf{T} \end{pmatrix}\boldsymbol{U}(\boldsymbol{x}) \succeq \boldsymbol{0}. \end{aligned} \tag{11}$$

where $\boldsymbol{U}(\boldsymbol{x})$ denotes the solution to (10).

Problem (11) is converted into an ordinary non-linear optimization problem (NLP) by noting that $\mathbb{S}_+^q \ni \boldsymbol{A} \Leftrightarrow \exists \boldsymbol{L} \in \mathcal{L}^q : \boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^\mathsf{T}$, where $\mathcal{L}^q$ denotes the set of lower triangular $q \times q$-matrices with non-negative diagonal entries. This leads to the problem

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathcal{H},\, z\in[0,1],\, \lambda\in[0,1],\, \boldsymbol{L}\in\mathcal{L}^{d+1}_{[-1,1]}} \quad z \\ &\text{subject to} \quad \begin{pmatrix} \lambda\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & z-\lambda \end{pmatrix} - \frac{1}{z_0}\begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{f}_0^\mathsf{T} \end{pmatrix}\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{L}\boldsymbol{L}^\mathsf{T}, \end{aligned} \tag{12}$$

where $\mathcal{L}^{d+1}_{[-1,1]}$ denotes the set of lower triangular $q \times q$-matrices with each diagonal entry in $[0, 1]$ and each off-diagonal entry in $[-1, 1]$. These bounds are motivated by the fact that

$$\begin{pmatrix} \lambda \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & z - \lambda \end{pmatrix} - \frac{1}{z_0} \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{f}_0^{\mathsf{T}} \end{pmatrix} \boldsymbol{U}(\boldsymbol{x}) \preceq \boldsymbol{I},$$

which holds since $\lambda \leq 1$ and $z \leq 1$ and the second matrix to the left is positive semi-definite [recall (8)]. At a feasible point then, $\boldsymbol{L}\boldsymbol{L}^{\mathsf{T}} \preceq \boldsymbol{I}$, showing that the diagonal elements of $\boldsymbol{L}\boldsymbol{L}^{\mathsf{T}} - \boldsymbol{I}$ must be non-positive. The latter implies the aforementioned bounds on the entries of $\boldsymbol{L}$.

A theoretical drawback of the NLP-formulation (12) is that it might have stationary points not present in problem (11). In practise we have however never experienced convergence to such points, and this theoretical drawback therefore outweighs practical drawbacks of e.g. PENLAB [11] which requires evaluation of second-order derivatives.

7. Numerical example

Figure 1 shows a numerical example. The ground structure is a cylinder, meshed with $m = 14640$ 8-node hexahedral elements, resulting in a total of $n = 46800$ displacement degrees of freedom. Problem (12) is solved with the

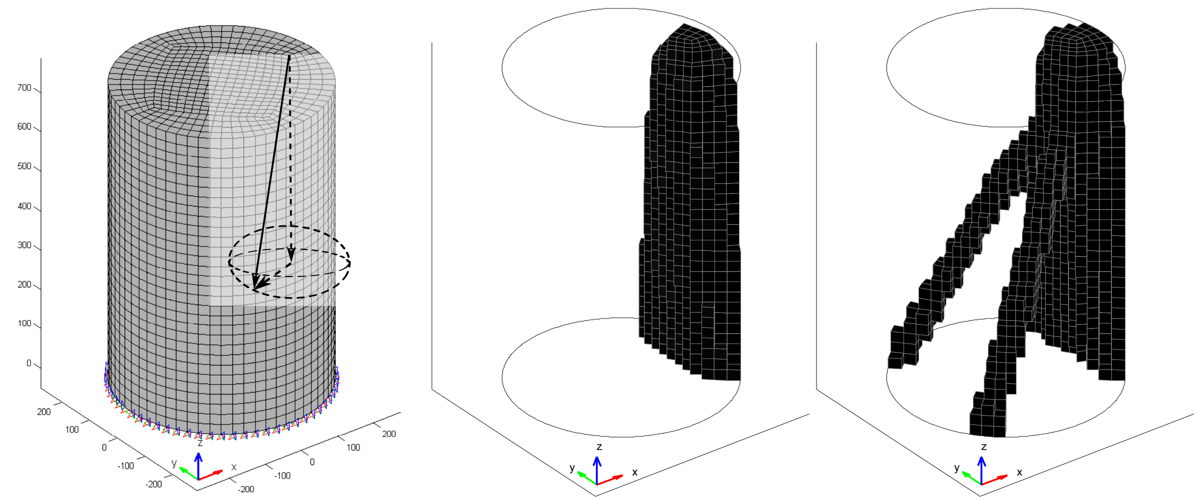

Figure 1: Numerical example. Left: Ground structure, with dimensions in [mm]. The figure shows the total load as the sum of a nominal load (dashed arrow in negative z-direction) and an uncertain part which varies in the sphere. Middle: Minimum compliance design for a single load case. Right: Worst-case compliance design.

code fminsdp for non-linear SDPs [18], using Ipopt [19] as NLP-solver. First-order derivatives of the non-linear part of the matrix constraint are computed using the adjoint method [9], and Ipopt's limited-memory quasi-Newton approximation, with 90 correction pairs, is used for the Hessian of the Lagrangian. The update strategy for the barrier parameter is set to "adaptive", and default settings used otherwise.

A load of 1000 [N] applied at the top of the structure pointing in the negative z-direction is used as the nominal load, and in the robust formulation we allow for variations of up to 100 [N] in each coordinate direction, i.e. the matrix $\boldsymbol{B} \in \mathbb{R}^{3 \times 46800}$ in (2) is given by $\boldsymbol{B} = (\boldsymbol{0}\ 100\boldsymbol{I}\ \boldsymbol{0})$, where $\boldsymbol{I}$ is a three-by-three identity matrix positioned at the degrees of freedoms associated with the loaded node.

The middle plot in Fig. 1 shows a design obtained when uncertainty is not accounted for; the result is a curved column. The right plot shows a designed obtained when taking load variations into account. Now the column is supported by two legs – intuitively a more robust design.

8. Concluding remarks

We have presented a problem formulation for deterministic worst-case compliance design in the form of a non-linear semi-definite program whose computational cost is similar to that of solving an ordinary minimum compliance problem with three (in 2D) or four (in 3D) load cases. The optimization problems are readily solved using software for ordinary NLPs.

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