Structural optimization of standardized trusses by dynamic grouping of modules

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1. Abstract

Modular structures tend to be widely used in civil engineering since components can be mass-produced in high quality controlled facilities, leading to economy of construction and improved reliability. Existing research in optimization of modular structures has focused primarily on the notion of modularity where only the topology is repeated. This paper presents a novel approach for dynamic grouping and topology optimization of modular structures, including the spatial orientation of the modules as an additional design variable. This extends the standard notion of modularity by accounting for the topology invariance of the module under rigid body rotations. Group theory is used to handle the spatial rotations and gives a straightforward and efficient mathematical representation of the module properties in terms of a permutation matrix for the rotation and continuous variables for the topology. Further theoretical developments are proposed to couple the existing dynamic grouping techniques for module linking, which is coupled with the proposed method of optimization for modular structures. The proposed approach is illustrated through an academic modular truss bridge, where a memetic algorithm is used to optimize simultaneously the topology and the orientation of the modules.

2. Keywords: Structural optimization, modular structures, group theory.

3. Introduction

Stronger requirements on sustainability, safety, and cost push architects and engineers to develop new philosophies of design. Among them, modularity (i.e., dividing a complex structure into simpler subsystems that can be mass-produced in high quality-controlled facilities [8]) offers substantial advantages. First, the repeatability of the components together with their independence allows for an industrial manufacturing of the modules, leading to better quality controls of production, higher safety measures, and a reduced sensitivity to weather interferences. Secondly, prefabrication of the modules leads to a significant time saving during construction since several tasks can be carried out in parallel, i.e., the geotechnical phases and the manufacturing of modules [11]. While modularity in construction is not new, optimization of modular structures is still largely unexplored and can lead to improved sustainability, safety, and economy.

The coupling between modularity and optimization involves two scales that must be considered concurrently. On one hand, the module scale is defined by the topology and the shape variables identically repeated throughout the structure. On the other hand, the structure scale describes the way the whole system is assembled starting from the initial module. Topology optimization for modular structures has been carried out for continuous and discrete structures but is still restricted to the module scale, without the capability of changing the module orientation [6, 1]. In other words, the module topology is repeated throughout the whole structure using standard symmetry operations like translation, rotation, and reflection. On a different standpoint, authors in [12] focused on the optimal orientation of fixed-topology modules for architectural purposes, i.e., building communication networks for pedestrian traffic. Starting from a fixed topology, a genetic algorithm evaluated the optimal module assembly to minimize the walking distance between two terminals. Finally, considering a small number of identical modules, so-called a group, results in added flexibility in the design and improves the optimization process, while keeping the economical and functional advantages of modularity [1].

From these observations, it follows that the optimization focuses either on the module scale or on the assembly process to build the whole structure, but always through a decoupled methodology. Therefore, the present paper proposes a unified formulation for optimizing modular truss structures assuming a linear elastic behaviour, focusing simultaneously on the topology of the modules, their spatial orientations, and the way they are dynamically grouped. This problem, which includes different types of design variables, is solved using a memetic algorithm: a simultaneous analysis and design technique is used for the topology optimization (to handle a large number constraints and variables) while elements of group theory efficiently integrate the module rotations.

4. Group theory and rigid body rotations

Including module rotation in the optimization requires some care related to the structure assembly and the non-commutative character of the rigid body rotations. This section addresses the numerical approach for the latter problem. By denoting $d \in \{2,3\}$ the spatial dimension and N_s the number of support reactions, the number of degrees of freedom is $N_d = d N_n - N_s$, with N_n being the number of nodes. The pin-joined modular structure is made of N_m identical modules, each of them composed of N_{em} elements. Due to topological repetition, all modules can be expressed through a reduced number of variables for the cross sections and the bar lengths i.e., $\mathbf{a} \in \mathbb{R}_+^{Nem}$ and $\mathbf{l} \in \mathbb{R}_+^{Nem}$ respectively.

Accounting for spatial rotations in modular structures is of great interest, since fixing the way the modules are assembled before carrying out the topology optimization will inherently bias the process [6]. Group theory provides a rigorous mathematical background to systematically describe symmetry operations, i.e., the operations that map an object into coincidence with itself. Widely used in chemistry and physics [5], extensions to civil engineering applications have been carried out to reduce the computational effort using a particular group, so-called the point group, that leaves one point fixed under operations. For spatial rotations, the special orthogonal group SO(3) is the most convenient set of symmetry operations to easily handle module orientation modifications while ensuring the topology invariance. Indeed, the latter corresponds to the group of all the rotations about the origin of a three-dimensional Euclidean space $\mathbb{R}^3$.

In structural analysis, rotating the modules in space using SO(3) requires some care: starting from an initial configuration, the module rotation leads to the dissociation between the forces and the nodes where they were initially applied, leading to unstable structures. Considering only a finite set, so-called admissible rotations, permits the circumvention of this problem of stability, by ensuring a perfect superposition between the initial and the final configurations for the boundary nodes only. Doing so, the symmetry operation will only consist of permuting the cross sections of the module elements in addition to continuously rotating the nodes inside the module. In group theory, any element of a symmetry group is associated a matrix representation. In the case of SO(3) for module rotation, one defines the matrix representation $\mathbf{P}$ such that, when rotating a module, a new topological configuration $\mathbf{a}'$ is reached and expressed by $\mathbf{a}' = \mathbf{P}\,\mathbf{a}$ [10]. The latter formulation presents attractive outcomes since it allows for a perfect decoupling between the rotation $\mathbf{P}$ and the topology $\mathbf{a}$, identical for all the modules. An illustration of the way the permutation matrices are acting on the structure is given in Fig. 1, where a rotation of 120 degrees is performed around the center c.

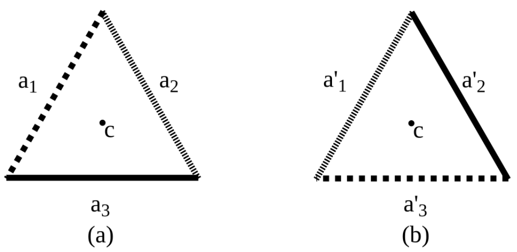
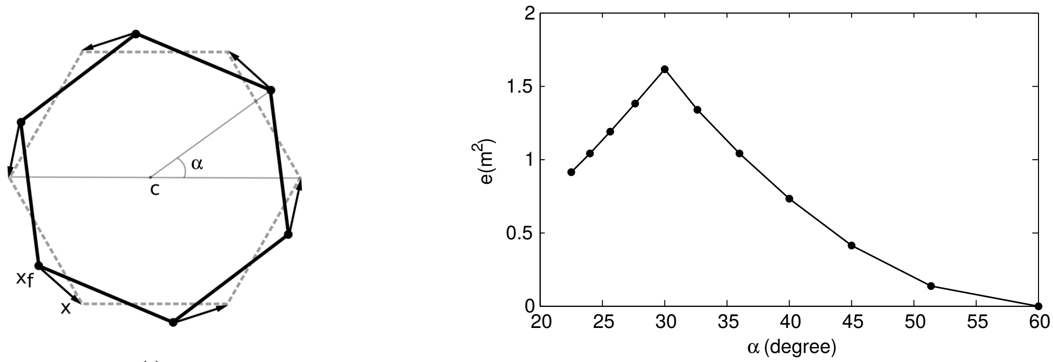


Figure 1: Effect of a permutation matrix on the topology of a triangular structure. The initial and final topologies, after a counter-clockwise rotation of 120 degrees around the point c, are depicted in (a) and (b) respectively. The final cross section vector $\mathbf{a}'$ is obtained by a change in the topology between the bars and expressed by $\mathbf{a}' = \mathbf{P}_c\,\mathbf{a}$.

Building the set of admissible permutation matrices can be performed using Euler's theorem, stating that any rotation around some axis is composed of a combination of three orthogonal rotations. To do so, one starts from the rotations matrices $\mathbf{P}_{nx}$, $\mathbf{P}_{ny}$ and $\mathbf{P}_{nz}$, corresponding to the matrix representations of the smallest admissible rotations of an angle $\alpha = 2\pi/n$ (with $n \in \mathbb{R}_0^+$), in the three orthogonal directions x, y and z respectively. Evaluating these three matrices is purely geometrical and consists of considering the module as a convex polygon that is rotated to ensure a perfect superposition between the initial and the final configuration. Such an operation can easily be carried out using a scalar measure, represented by the total square Euclidean distance e, that becomes zero for any admissible rotation:

$$\mathrm{e}_n = \sum_{i=1}^{n_b} \min_j ||\mathbf{x}_{f,i}(n) - \mathbf{x}_j||^2 \tag{1}$$

where n_b denotes the number of boundary nodes per module, $\mathbf{x}_{f,i}(n)$ and $\mathbf{x}_j$ the i^{th} final and the j^{th} initial position of the boundary nodes respectively, for an angle of rotation $\alpha = 2\pi/n$. An example of the evolution of the total Euclidean distance with respect to α is illustrated in Fig. 2 for a hexagonal module, under a rotation around the c



(a)

Figure 2: The rotation of an hexagonal module around the c axis, with it initial and final configuration in gray and black respectively (a). The total square Euclidean distance, obtained by summing the square of the norm of the black vectors, evolves towards zero for an angle of rotation ($\alpha = 2\pi/n$) of 60 degrees (b).

axis. Retaining the way the nodes are permuted when the total Euclidean distance is zero enables the building of the permutation matrix: $\mathbf{P}_{kp}$ takes a unitary value if the k^{th} and the p^{th} bars are permuted.

The major problem when handling spatial rotations using SO(3) relies on its non-commutative characteristic i.e., $\mathbf{P}_{nx}\mathbf{P}_{ny}\mathbf{P}_{nz} \neq \mathbf{P}_{ny}\mathbf{P}_{nz}\mathbf{P}_{nx}$ [5]. As a consequence, the order of the matrix multiplication matters for being able to explore all of the possible module orientations, which unfavourably impacts the size of the design space. Well known techniques have been developed to deal with combinatorial optimization and permutation-based problems without exploring the whole design space. However, most of them require sufficient information on the problem to enumerate the solutions in a clever way, by disregarding subsets of the design space that do not contain the solution [2]. In the present study, the non-commutative and discrete character of the design space for the admissible rotations forbids such an approach, leading to a standard complete enumeration procedure:

$$\mathbf{P}_{(L_\xi, L_\eta, L_\zeta)} = \mathbf{P}_{n_\xi}^{L_\xi}\mathbf{P}_{n_\eta}^{L_\eta}\mathbf{P}_{n_\zeta}^{L_\zeta} \quad \begin{cases} L_\xi = 1, \dots, n_\xi - 1 \\ L_\eta = 1, \dots, n_\eta - 1 \\ L_\zeta = 1, \dots, n_\zeta - 1 \end{cases} \qquad \forall \xi, \eta, \zeta \in \{x, y, z\} \tag{2}$$

where $\mathbf{P}_{n_\xi}^{L_\xi}$ corresponds to a rotation in the ξ^{th} direction with an angle of $2\pi L_\xi / n_\xi$. Since equation (2) generates the complete set of admissible rotations, the computational complexity becomes very high due to its exponential dependence with respect to the number of modules. Considering a structure made of N_m modules where each of them has, in average, n possibilities of rotation in each direction. The total number of possible combinations, accounting for the non-commutativity of the rotations, is exactly $3!\,n^{dN_m}$, leading to a time complexity $T(n, N_m) = \mathcal{O}(n^{N_m})$. Fortunately, in the case of module rotations, different combinations of permutation matrices lead to the same final configuration, thereby greatly reducing the complexity of the problem. In the hexagonal example in Fig. 2, a reduction of 12^{N_m} times the initial space of search can be performed, leading to a problem that is numerically tractable without extensive computational efforts.

5. Hybrid optimization algorithm for structural optimization of modular truss structures

The optimization problem addressed in the present paper consists of minimizing the compliance (the compliance is a measure of the energy stored in a structure undergoing deformation) of a modular truss structure under (i) the static and kinematic equilibrium conditions and (ii) a limitation on the allowable final volume and nodal displacements. First, modifications are made to the equilibrium equations to account for the permutation matrices and the topological repetition in the modular structures.

5.1. Equilibrium equations for modular truss structures

The module cross section vector $\mathbf{a}_m$ can be expressed, according to group theory, by a permutation matrix $\mathbf{P}^m$ and the fundamental cross section $\mathbf{a}$. Incorporating the latter formulation into the kinematic and static compatibility equations [9] enables the expression of the classical equilibrium equations for modular structures:

$$\sum_{m=1}^{N_m}\sum_{e=1}^{N_{em}} \frac{E\,a_{em}}{l_{em}} \gamma_{em}^T \gamma_{em}\,\mathbf{u} = \mathbf{f} \tag{3}$$

where a_{em} denotes the cross section of the e^{th} element in the m^{th} module, γ_{em} represents its direction cosines and

$\mathbf{u} \in \mathbb{R}^{N_d}$ and $\mathbf{f} \in \mathbb{R}^{N_d}$ denotes the displacement field and external force vector respectively. By introducing the permutation matrices $\mathbf{P}$ and the cross section vector $\mathbf{a}$, the module cross section becomes

$$a_{em} = \sum_{r=1}^{N_{em}} \mathbf{P}_{er}^{m}\, a_r \tag{4}$$

where $\mathbf{P}^m$ denotes the permutation matrix of the m^{th} module. Equations (3) and (4) can be coupled together to explicitly express the equilibrium equations in terms of a module orientation $\mathbf{P}^m$ and the topology information $\mathbf{a}$ (identical for all the modules) in the optimization problem.

5.2. Problem formulation for optimization of modular truss structures

Typically, the numerical optimization and analysis phases are considered distinct in a computational sense. Given a design domain, the structure is solved exactly through a finite element analysis, hence only the optimization is involved in the design space related to the cross section vector $\mathbf{a}$, distinct from the state space defined by the displacement field $\mathbf{u}$. This procedure, so-called nested analysis and design, is solved by an iterative procedure until the optimum is reached. Another paradigm, namely the simultaneous analysis and design, treats the design and state variables independently, so that equilibrium equations (3) are set as equality constraints and do not need to be solved at each iteration [4, 3]. Following the latter formulation, coupled with equations (3) and (4), the problem definition for compliance minimization of modular structures can be stated as

$$\begin{aligned}
&\underset{\mathbf{P},\mathbf{a},\mathbf{u}}{\text{Minimize}} \frac{1}{2} f^T \mathbf{u} \\
&\quad \text{s.}t. \sum_{m=1}^{N_m} \sum_{e=1}^{N_{em}} \frac{E\, a_{em}}{l_{em}} \gamma_{em}^T \gamma_{em}\, \mathbf{u} = \mathbf{f} \\
&\quad |u_{e,m}| - u_{\lim} \leq 0 \\
&\quad N_m \sum_{e=1}^{N_{em}} a_e\, l_e - v_{\lim} = 0
\end{aligned} \tag{5}$$

with $\mathbf{a} \in \mathbb{R}_{+}^{N_{em}}$, $\mathbf{P} \in \mathbb{P}$, the set of admissible permutation matrices $\mathbb{P} \subset \mathbb{R}^{N_{em} \times N_{em}}$. In equation (5), $u_{\lim}$ and $v_{\lim}$ correspond to the allowable magnitudes for the displacements and the volume respectively. The major issue in solving equation (5) is the different nature of the design variables. Although simultaneous analysis and design topology optimization problem can be efficiently solved using mathematical programming, the discrete character of the permutation matrices makes the resolution of equation (5) difficult with a single algorithm.

Among algorithms that are able to manage multiple types of design variables, hybrid algorithms offer a great potential to efficiently solve equation (5). Indeed, combining global and a local search techniques provides the advantages of both methods, i.e., being able to explore the design space with non-continuous variables and easily handling large scale problems [7]. In this study, the Lamarckian approach is used, in which a genetic algorithm and a gradient-based method are employed together [7]. In such an approach, the design variables are divided into two parts, each of them are handled by one of the two algorithms. The genetic algorithm manages the discrete variables for the permutation matrices while the gradient-based method works only with the continuous variables, and both are coupled for the evaluation of the objective functions and constraints. The general functioning of the proposed memetic algorithm is depicted in Fig. 3: in one iteration of the algorithm, the genetic operators perform modifications to the permutation matrices before applying the interior-point algorithm, to drive the individuals to local optima for a fixed module orientation. The major advantage of the proposed approach relies in the use of a mathematical programming to solve equation (5) for fixed values of the permutation matrices, taking advantage of the inherent sparsity of the problem as well as the capability of providing analytically the Jacobian matrices [4].

6. Numerical application

To illustrate the proposed approach, an academic application of a simply supported bridge truss made of 4 identical modules is studied (Fig. 4). The design domain is made of a 13 x 3 grid regularly spaced by one meter while the ground structure at the module scale is composed of 66 bars. The bridge, made of steel (S235), has a span of 12 meters and is submitted to a single load acting at the upper node at midspan with a magnitude of 100 kN. The Young's modulus E and the limitations on the volume $v_{\lim}$ and the displacement $u_{\lim}$ are equal to 235 MPa, 10% of the initial volume of the ground structure, and $1/800$ of the span respectively. To solve the non-linear problem (5), a genetic algorithm for integer programming handles the permutation matrices (using truncated Laplacian crossover and power mutation operators), while an interior-point algorithm with analytic Jacobian handles the cross sections

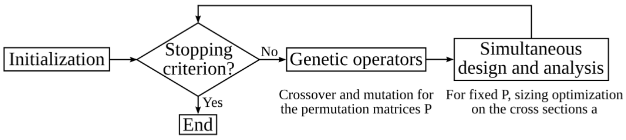


Figure 3: General flowchart of the memetic algorithm. The permutation matrices are managed by the genetic operators while the cross sections **a** and the displacement field **u** are handled by the local search algorithm

and the displacement field. A small population size of 20 individuals is sufficient to ensure a fast convergence of the algorithm since only the spatial orientation variables are handled by the genetic algorithm. The convergence criterion is based on the stable condition (10^{-8}) for the design variables and the objective function.

The proposed approach is compared with the optimization of periodic modular structures [6]. In the latter, the module orientation is a priori fixed, taking advantage of the structural symmetry; in the present example, this corresponds to a topology symmetry between the modules 1 and 4 and the modules 2 and 3. The results are depicted in Fig. 5, corresponding to the module topology and their spatial orientations to build the whole structure, where a final compliance of 25.74 J is reached. Fig. 6 gives the results obtained by the proposed approach, where a gain in efficiency is clearly demonstrated in comparison with the previous method: simultaneous module topology and spatial orientation decreases the objective function to 22.49 J. The cross sections as well as the module orientations are such that the whole structure presents strong elements to handle the flexural loads on the bridge. In addition, it can be noticed that the latter solid configuration exhibits similarities with those obtained from continuous optimization for a simply supported bridge [3].

From a computational viewpoint, 30 generations are sufficient to respect the stopping criterion. This fast convergence towards the solution can be explained by the simultaneous optimization of the module orientation and their topology. The same optimal configuration can be attained by different module topologies and spatial arrangements, representing the existence of multiple but equivalent local minima. In addition, the latter observation also explained the high robustness of the algorithm with respect to the genetic parameters, where variations on the results were only observed for a population size less than 10 individuals together with a convergence tolerance lower than 10^{-3}.

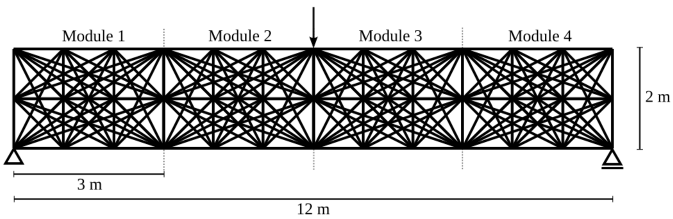


Figure 4: Ground structure

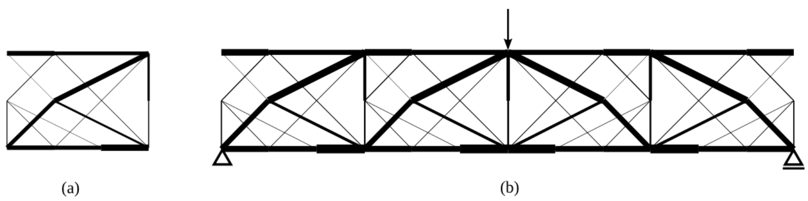


Figure 5: The results obtained when imposing the topology symmetry, with the module topology (a) and the final module distribution (b) (compliance = 25.74 J)

6. Conclusions

This paper describes a novel approach for the structural optimization of modular structures i.e., structures made of identical topological components. By coupling elements of group theory and mathematical programming techniques, the problem is properly addressed and solved through a memetic algorithm, allowing for a clear decoupling between design variables of different natures. A genetic algorithm handles the discrete variables using truncated

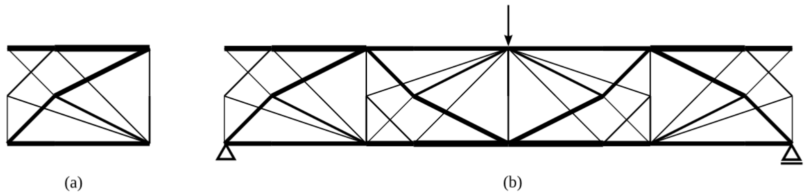


Figure 6: The results obtained by the proposed approach, with the module topology (a) and the final module distribution (b) (compliance = 22.49 J)

genetic operators while, for fixed module orientations, a simultaneous analysis and design manages the topology optimization at the module scale; the latter barrier problem is solved using an interior-point algorithm, which is well suited for problems involving a large number of design variables and constraints. With a limited computational effort, optimal configurations, in terms of module orientation and topology, are generated and provide better optimization results than applying standard optimization approaches for modular structures. For future works, extensions of the proposed method to manage partial modularity and dynamic grouping of modules will be investigated, in order to provide a complete unified framework.

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7. References

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