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EVOLUTIONARY SHAPE OPTIMIZATION FOR STRESS MINIMIZATION

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Introduction

Stress minimization is a major aspect of structural optimization in a wide range of engineering designs. In classical finite element and sensitivity based shape design methods, the nodal coordinates are treated as the design variables[1,2]. To accommodate the continuous variations of nodal coordinates, a re-meshing process usually needs to be employed. To avoid the time consuming re-meshing process, various fixed grid methods have been developed in recent years. However, there exist two limitations in these methods. Firstly, these methods follow the continuous variable strategy (e.g. taking element densities as design variables) in order to use conventional mathematical programming. This needs to involve complicated mathematical operations and calculations. Moreover, there exists difficulty in dealing with the problem that the location of a maximum stress changes as the design optimization progresses. Secondly, these methods exclusively employed stiffness considerations [3], where the mean compliance is formulated as the objective while the weight or volume of the structure is posed as a constraint [3,4]. This cannot ensure higher reliability and durability, if stress itself is not considered in the design process [5-7].

Realizing the aforementioned facts, this paper develops a stress based evolutionary structural optimization method (ESO), in which the discrete variable method with the binary decision-making is used to decide the element's presence or absence [8-11]. On the basis of finite element analysis, a stress sensitivity number is derived to estimate the stress change due to element removal or addition. Following the evolutionary optimization procedure [8, 10], an optimal design with a minimum maximum stress is achieved by gradually removing or adding those elements which have the lowest or highest stress sensitivity numbers respectively. A classical example of fillet design presented here demonstrates the capacity of the proposed method for solving stress minimization problems.

Sensitivity Analysis

In the evolutionary structural optimization method, a structure can be optimized by removing or adding elements. That is to say that, the element itself, rather than its associated physical parameters, is treated as the design variable. For this reason, the sensitivity analysis presented here is derived with respect to the absence or presence of an entire element.

Displacement sensitivity number. In finite element analysis (FEA), the static behavior of a structure is represented by the equilibrium equation

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{p}, \qquad (1)$$

where $\boldsymbol{K}$ is the global stiffness matrix of the structural system, $\boldsymbol{u}$ is the global nodal displacement vector, $\boldsymbol{p}$ is the nodal load vector. From Eq. (1), it is easy to find that removal or addition of any element results in the variation in displacement vector $\boldsymbol{u}$ of the structure by

$$\Delta\boldsymbol{u} = -\boldsymbol{K}^{-1}\cdot\Delta\boldsymbol{K}_i\cdot\boldsymbol{u}, \qquad (2)$$

where $\Delta\boldsymbol{K}_i$ denotes the variation in global stiffness matrix due to removing or adding the ith element. To find the change in a specified jth displacement component u_j, a unit virtual load vector $\boldsymbol{f}_j$ corresponding to u_j is introduced. Multiplying Eq. (2) by $\boldsymbol{f}_j$, we obtain

$$\Delta u_j = \boldsymbol{f}_j^{\mathrm{T}}\cdot\Delta\boldsymbol{u} = -\boldsymbol{f}_j^{\mathrm{T}}\boldsymbol{K}^{-1}\Delta\boldsymbol{K}_i\boldsymbol{u} = -\boldsymbol{u}_j^{\mathrm{T}}\Delta\boldsymbol{K}_i\boldsymbol{u}, \qquad (3)$$

where $\boldsymbol{u}_j$ represents the solution of Eq. (1) under the virtual load $\boldsymbol{f}_j$ (i.e. $\boldsymbol{K}\cdot\boldsymbol{u}_j = \boldsymbol{f}_j$). The matrix calculation of Eq. (3) can be carried out at element level. Thus the value

$$\alpha_{ij} = \pm\boldsymbol{u}_{ij}^{\mathrm{T}}\boldsymbol{K}_i\boldsymbol{u}_i \qquad (4)$$

is defined as the *displacement sensitivity number*, which is used to estimate the displacement change of the jth degree of freedom due to the removal or addition of the ith element, where the + and − signs are for removed and added element respectively, $\boldsymbol{K}_i$ is the element stiffness matrix of element i, $\boldsymbol{u}_i$ and $\boldsymbol{u}_{ij}$ are the element displacement entries due to the real and virtual loads respectively.

Stress sensitivity number. In finite element analysis, the stress vector σ of the kth element can be calculated on the basis of its nodal displacement vector $\boldsymbol{u}^k$ by

$$\sigma = \boldsymbol{D}\,\boldsymbol{B}\,\boldsymbol{u}^k, \qquad (5)$$

where $\boldsymbol{D}$ and $\boldsymbol{B}$ represent the conventional elastic and strain matrices respectively. By combining the displacement sensitivity numbers, the change of the kth element's stress vector due to removing or adding the ith element can be found as

$$\Delta\sigma = \boldsymbol{D}\ \boldsymbol{B}\ \Delta\boldsymbol{u}^k = \boldsymbol{DB}\left\{\Delta u_1, \Delta u_2, \cdots, \Delta u_j, \cdots, \Delta u_n\right\}_k^{\mathrm{T}} = \boldsymbol{D}\ \boldsymbol{B}\ \alpha\,, \tag{6}$$

where $\alpha = \left\{\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ij}, \cdots, \alpha_{in}\right\}_k^{\mathrm{T}}$ is referred to as displacement sensitivity vector of element k, n represents the number of degrees of freedom in element k. Using Eq. (6), the change of the kth element stress can be calculated on the basis of the calculations for its all displacement component variations.

In engineering design, the stress level at each element can be measured by some sort of average of all the stress components. Taking 2D isotropic elasticity as an example, the von Mises stress is frequently used, i.e.

$$\bar{\sigma} = \sigma_{vm} = f\left(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\right) = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2}\,. \tag{7}$$

Without losing generality, the variation in element's stress function can be found by

$$\Delta\bar{\sigma} = \nabla f^{\mathrm{T}}\ \Delta\sigma = \nabla f^{\mathrm{T}} \cdot \boldsymbol{D}\ \boldsymbol{B}\ \alpha = \gamma \cdot \alpha\,, \tag{8}$$

where $\nabla f = \left\{\partial f/\partial\sigma_{xx}, \partial f/\partial\sigma_{yy}, \partial f/\partial\sigma_{xy}\right\}_k^{\mathrm{T}}$ represents the gradient vector of the stress function and $\gamma = \nabla f^{\mathrm{T}}\boldsymbol{D}\ \boldsymbol{B} = \left\{\gamma_1, \gamma_2, \cdots, \gamma_j, \cdots, \gamma_n\right\}_k^{\mathrm{T}}$. Substituting Eq. (4) into (8) gives

$$\Delta\bar{\sigma} = \pm\sum_{j=1}^{n}\left(\gamma_j \boldsymbol{u}_{ij}^{\mathrm{T}} \boldsymbol{K}_i \boldsymbol{u}_i\right) = \pm\sum_{j=1}^{n}\left(\gamma_j \boldsymbol{u}_{ij}^{\mathrm{T}}\right) \cdot \boldsymbol{K}_i \cdot \boldsymbol{u}_i\,. \tag{9}$$

It is noted that $\boldsymbol{u}_{ij}^{\mathrm{T}}$ $(j = 1, 2, \cdots, n)$ is the solution of the jth virtual system of $\boldsymbol{K} \cdot \boldsymbol{u}_{ij} = \boldsymbol{f}_j$. Multiplying γ_j to this and then adding all such n virtual equilibrium equations, we have

$$\boldsymbol{K} \cdot \sum_{j=1}^{n}\left(\gamma_j \boldsymbol{u}_{ij}\right) = \sum_{j=1}^{n}\gamma_j \boldsymbol{f}_j\,, \qquad \text{or:} \qquad \boldsymbol{K} \cdot \tilde{\boldsymbol{u}}_{ik} = \tilde{\boldsymbol{f}}\,, \tag{10}$$

where $\tilde{\boldsymbol{u}}_{ik} = \sum_{j=1}^{n}\gamma_j \boldsymbol{u}_{ij}$ represents a virtual displacement vector and $\tilde{\boldsymbol{f}} = \sum_{j=1}^{n}\gamma_j \boldsymbol{f}_j$ denotes a virtual load. In other words, Eq. (10) introduces a new virtual system where n non-zero components of the virtual load vector $\tilde{\boldsymbol{f}}$ are respectively the corresponding coefficients γ_j. Thus, the solution $\tilde{\boldsymbol{u}}_{ik}$ provided by the new virtual system (10) has

$$\alpha_{\sigma,i} = \Delta\bar{\sigma} = \pm\ \tilde{\boldsymbol{u}}_{ik}^{\mathrm{T}} \cdot \boldsymbol{K}_i \cdot \boldsymbol{u}_i \tag{11}$$

defined as the ith element's *stress sensitivity number*, which is used to indicate the stress change of element k due to the absence or presence of element i. Note that Eq. (11) for stress sensitivity analysis is similar to Eq. (4) for displacement sensitivity analysis. In fact, the two types of sensitivity numbers are calculated in exactly the same way, with a difference only in their respective virtual load vectors.

It is worth pointing out that the stress sensitivity number $\alpha_{\sigma,i}$ can be positive or negative, which implies that the stress of element k may increase or decrease when elements are removed or added. This provides a means of controlling the stress of a specified element by appropriately removing or adding elements in a structure.

Evolutionary Optimization Procedure and Implementation

In finite element analysis, the element absence or presence can be simply represented by a property of type 0 or 1. After subdividing the design domain with a dense mesh, progressive removal or addition of elements from the initial structure presents two opposite direction evolutionary processes. The former starts from a status with a fully populated domain (the property type of all elements is set to 1) while the latter starts from a status with the least design domain (all equal to 0 except for those elements for connecting the loads to supports). In the latter case, although the elements are stored in the FEA database, they are not included in the analysis. This means that the solution may not provide the full information on the element displacement entries $\boldsymbol{u}_i$ and $\boldsymbol{u}_{ij}$, which are required in the calculation of the stress sensitivity numbers. Considering the fact that element additions are always carried out by attaching the candidate elements (whose property type = 0) onto the boundary elements (whose property type = 1), it is possible to form a fictitious displacement field by an appropriate extrapolation technique, where the unknown displacement components of the candidate element are found from the known components of its corresponding attached elements. As a result, all necessary information for the stress sensitivity analysis is obtained.

To follow the typical ESO procedure [8], the stress sensitivity numbers of all elements are translated to positive values. This can be accomplished by simply adding a certain positive constant number C which is large enough to have all sensitivity numbers greater than zero i.e. $\hat{\alpha}_{\sigma,i} = \alpha_{\sigma,i} + C \geq 0$. The evolutionary criterion for element removal or addition is determined by comparing the stress sensitivity number with the highest value, i.e. if it satisfies $\hat{\alpha}_{\sigma,i} \leq RR_{SS} \times \hat{\alpha}_{\sigma,max}$, the element is removed or if it satisfies $\hat{\alpha}_{\sigma,i} \geq (1 - RR_{SS}) \times \hat{\alpha}_{\sigma,max}$, the element is added, where RR_{SS} is called *Rejection Ratio* for element removal while $(1 - RR_{SS})$ is called *Inclusion Ratio* for element addition. The process of the element removal or addition is repeated using the same value of RR_{SS} until a *Steady State* is reached, which means that there are no more elements being deleted or added at the current iteration. At this stage an *Evolutionary Rate* (*ER*) is introduced so that $RR_{SS+1} = RR_{SS} + ER$. A typical value for ER is around 0.1% to 1% to ensure a smooth change between two steady states. With the increased rejection ratio or the decreased inclusion ratio the cycle of finite element analysis and element removal or addition takes place again until a new steady state is reached.

The evolutionary iteration procedure for minimizing the maximum stress is given in detail as follows:

Step 1: Discretize the structure using a dense FE mesh, assign the initial property values of elements in the design domain to 1 or 0 for removing or adding ESO respectively, and define ESO driving parameter ER and set $RR_o = 0$;

Step 2: Perform a FEA for the real system, determine the maximum stress element k, and apply a virtual load $\tilde{f}$ as given in Eq.(10) onto this element;

Step 3: Perform another FEA for the virtual systems;

Step 4: Calculate stress sensitivity number $\alpha_{\sigma,i}$ using Eq.(11);

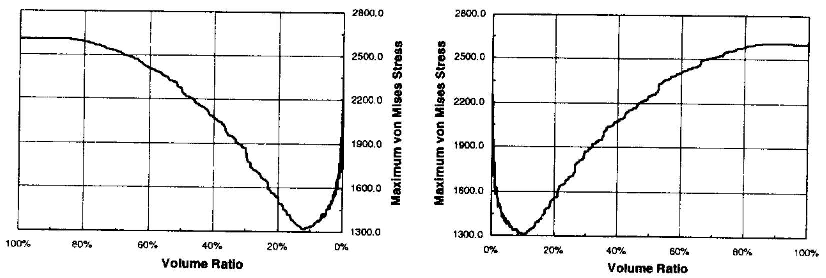
Step 5: Remove or add a number of elements with the lowest or highest $\hat{\alpha}_{\sigma,i}$ through criterion: $\hat{\alpha}_{\sigma,i} \leq RR_{SS} \times \hat{\alpha}_{\sigma,max}$ or $\hat{\alpha}_{\sigma,i} \geq (1 - RR_{SS}) \times \hat{\alpha}_{\sigma,max}$;

Step 6: If a *steady state* is reached, increase RR_{SS} by ER, i.e. $RR_{SS+1} = RR_{SS} + ER$ and set *SS*=*SS*+1, repeats Step 5; Otherwise, repeat Steps 2 to 5 until a minimized maximum stress is found.

Demonstrative Example: Fillet Design

The capabilities of the proposed evolutionary optimization procedure are demonstrated through a classical example of the fillet design, which is a well-known shape optimization problem investigated by many researchers [1,2,10]. In the finite element model, four node plane stress elements are used and only the top half of the structure is considered due to the symmetry. The region in the design domain $L \times H$ (here $L/H = 2$) is subdivided using a denser finite element mesh. An evolutionary rate (*ER*) of 0.1% is set for both element removal and addition procedures.

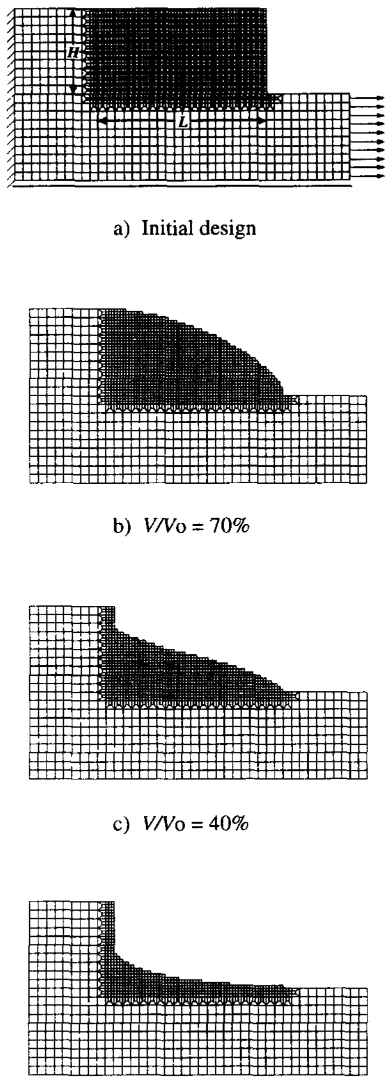
Fig. 1 shows the evolution histories of the maximum von Mises stresses in the fillet structure. A good symmetry can be observed from the two curves due to the opposition of volume changes in the two different procedures. The minimum stress points for both cases can be clearly identified at the volume ratio of around 12% (ratio of current structure to full one). To achieve this, the procedure with element addition seems much faster than that with element removal in this specific example.



a) ESO with element removal b) ESO with element addition

Fig. 1 Evolution histories of the maximum von Mises stresses

Figs. 2 and 3 show the evolution processes at several different volume ratios for the cases of element removal and element addition respectively. A good resemblance between the resulting shapes of the two inverse procedures can be clearly seen from the corresponding volume ratios. It is evident that the optimized results at volume ratio of around 12% are in good agreement with the classical solutions based on the continuous variable methods [1,2].



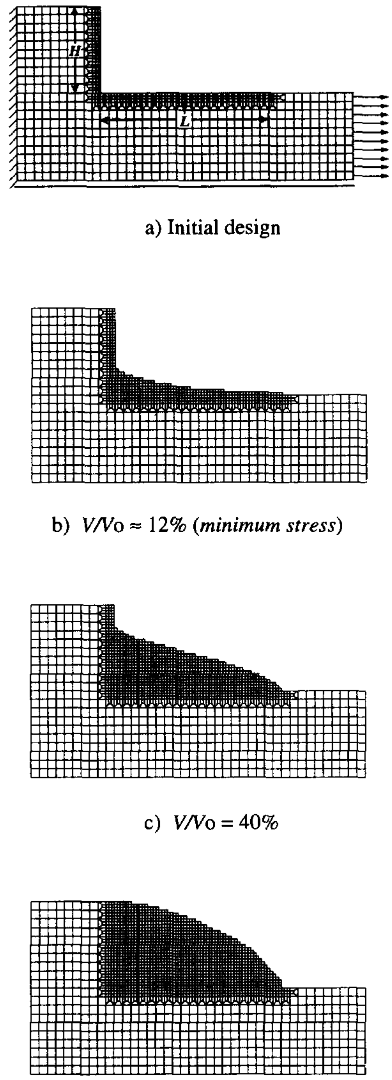
a) Initial design

b) $V/Vo = 70\%$

c) $V/Vo = 40\%$

d) $V/Vo \approx 12\%$ (*minimum stress*)

Fig. 2 ESO with element removal



a) Initial design

b) $V/Vo \approx 12\%$ (*minimum stress*)

c) $V/Vo = 40\%$

d) $V/Vo = 70\%$

Fig. 3 ESO with element addition

Concluding Remarks

In this study a stress sensitivity number with respect to element absence or presence is proposed. By introducing an adjoint virtual system, the calculation of the stress sensitivity number does not increase too much computational cost. According to the negative or positive values of the stress sensitivity number, a structure can be divided into two regions. Removing elements in the negative sensitivity region or adding elements in the positive sensitivity region will lead to a stress reduction in a specific element.

The evolutionary procedures for removing and adding elements are presented and compared in this paper. The symmetry of two approaches is demonstrated through a classical example. The optimization based on the proposed evolution criterion of stress sensitivity number offers a new capability for structural design aimed at minimizing the maximum stress. One of the distinct advantages of the presented method over the classical ones is that it does not need to involve complicated mathematical operations and programming, nor does it need to know more details than element stiffness matrix of FEA programs. This provides engineers with a very simple and practical alternative.

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