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Evolutionary structural optimization for stress minimization problems by discrete thickness design

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Abstract

Stress minimization is a major aspect of structural optimization in a wide range of engineering designs. This paper presents a new evolutionary criterion for the problems of variable thickness design whilst minimizing the maximum stress in a structure. On the basis of finite element analysis, a stress sensitivity number is derived to estimate the stress change in an element due to varying the thickness of other elements. Following the evolutionary optimization procedure, an optimal design with a minimum maximum stress is achieved by gradually removing material from those elements, which have the lowest stress sensitivity number or adding material onto those elements, which have the highest stress sensitivity number. The numerical examples presented in this paper demonstrate the capacity of the proposed method for solving stress minimization problems. The results based on the stress criterion are compared with traditional ones based on a stiffness criterion, and an optimization scheme based on the combination of both the stress minimization and the stiffness maximization criteria is presented.

Keywords: Finite element analysis; Stress minimization; Sensitivity analysis; Evolutionary structural optimization; Discrete thickness design

1. Introduction

The focus of many engineering design analyses is to find the maximum stress present in a given structure under all loading conditions. Usually, this maximum provides the basis of a design limit and is thus employed to determine the structural material and its weight. For this reason, stress minimization has always been a major concern of design engineers. This classic problem has attracted wide attention for many years, but previous work has mainly been limited to the reduction of stress concentration for such structural elements as fillets and holes by locally re-shaping the initial design [1,2].

In recent years, structural optimization with stiffness considerations has been exhaustively studied, as in Refs. [3–10]. In these publications, the mean compliance is usually formulated as the objective function while the weight or volume of the structure is posed as a constraint. However, such a design process cannot always warrant higher reliability and durability, if stress itself is not considered in the design process [11]. Usually, stress is presented in a form of constraints rather than an objective in the optimization processes [12–20]. Although this may guarantee the stress within a prescribed constraint, it cannot always make the stress minimum. As pointed out by Yang and Chen [11], when using the conventional finite element and mathematical programming based approaches, two major difficulties are encountered in the treatment of the stress localization and its high non-linearity with respect to the design variables. For these reasons, various continuous stress

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functions, such as the well-known Kreisselmeier–Steinhauser function [11], need to be employed to construct an objective or pose a constraint. In addition, to deal with the singularity in mathematical programming procedures of the stress based problems, appropriate relaxation techniques have to be used [14,19,20].

To avoid the adoption of the artificial stress functions and use of non-linear programming algorithms, in this study, the rule based evolutionary structural optimization (ESO) algorithm [6,21] is employed to solve such stress-based problems. The ESO method has been proposed and developed to simplify traditional structural optimization procedure. It is based on a simple concept that by systematically removing the material from the least efficient regions or adding the material onto the most efficient regions, the resulting structure evolves towards an optimum. Originally, the ESO method takes elemental stress level as the optimality criterion [21], in which the lowly (highly) stressed material is regarded as less (more) efficient or under (over) utilized from a standpoint of iso-strength design. As the material is redistributed, the stress levels in the remaining structure become more uniform or the efficiencies of material usage get more even. Recent studies by Mckeown [22] and the authors [23] have shown that such a fully stressed design, in fact, is equivalent to a stiffest design. In other words, although the optimization results in each part of structure to carry near the same levels of stress, it cannot always minimize the highest stress in the structure.

To achieve a design of stress minimization, it is essential to estimate the stress change in a specified element due to the variation of material allocation of the other elements. For this purpose, a *stress sensitivity number* is introduced in this paper. In the presented evolutionary optimization process, discrete thickness of elements is considered as design variables. In terms of the stress sensitivity number, the thicknesses of candidate elements are progressively redistributed so that the maximum stress is gradually minimized.

It should be pointed out that the proposed stress minimization criterion differs from the conventional stiffness maximization one. In these two criteria, the maximum stress is considered as an objective function for the former and could be treated as a constraint for the latter. Note that, in most of the typical stiffness designs, increases in maximum stress can be relatively small, and usually, do not violate the stress constraint [6]. Therefore, the stiffness maximization criterion is most commonly used, in which the strength requirement is reflected by means of the fully stressed or iso-strength concept [21,23,24]. It is believed that an ideal optimization should comply with both the stress minimization and the stiffness maximization. Under most circumstances, however, this is very difficult and an appropriate trade-off is usually required. For this reason, the paper also develops a weighting factor scheme to address this issue.

2. Sensitivity analysis

In variable thickness design, one of the common approaches of changing the material distribution of a structure is to reduce or increase thickness of some elements [6,8,25,26]. This can be done by simply assigning the element thickness to the next available one from a given set of discrete values. It is a desirable technical goal that such a process of material morphing leads to the reduction of the maximum stress in a structure. For this purpose, it is necessary to evaluate the contribution of elemental thickness variation to the reduction of the maximum stress, prior to the material redistribution.

2.1. Displacement sensitivity number

Suppose the ith element is to be re-sized to the next available lower thickness. This results in the change in the global stiffness matrix by $\Delta\hat{\boldsymbol{K}}_i = \boldsymbol{K} - \boldsymbol{K}_i^{\text{new}}$. The change in displacements can be determined by considering the equilibrium conditions before the change [6–10], i.e.

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{p} \tag{1}$$

and after the change, i.e.

$$(\boldsymbol{K} - \Delta\hat{\boldsymbol{K}}_i)\cdot(\boldsymbol{u} + \Delta\boldsymbol{u}) = \boldsymbol{p}, \tag{2}$$

where $\boldsymbol{K}$ denotes the global stiffness matrix of the old system, $\boldsymbol{u}$ the global nodal displacement vector, $\boldsymbol{p}$ the nodal load vector and $\Delta\boldsymbol{u}$ the change of the displacement vector $\boldsymbol{u}$. No change in the nodal load vector is assumed [7–10]. By subtracting Eq. (1) from Eq. (2) and ignoring the higher order term [10], the change in the displacement vector can be found as

$$\Delta\boldsymbol{u} = \boldsymbol{K}^{-1}\cdot\Delta\hat{\boldsymbol{K}}_i\cdot\boldsymbol{u}. \tag{3}$$

To find the change in a specified jth displacement component u_j, a virtual unit load vector $\boldsymbol{f}_j$, in which the jth component is equal to unity and all the others are equal to zero, is introduced. Multiplying Eq. (3) by $\boldsymbol{f}_j$, the change Δu_j in the specified jth displacement component due to the thickness change in the ith element, is determined by

$$\Delta u_j = \boldsymbol{f}_j^{\mathrm{T}}\cdot\Delta\boldsymbol{u} = \boldsymbol{f}_j^{\mathrm{T}}\boldsymbol{K}^{-1}\Delta\hat{\boldsymbol{K}}_i\boldsymbol{u} = \boldsymbol{u}_j^{\mathrm{T}}\Delta\hat{\boldsymbol{K}}_i\boldsymbol{u}, \tag{4}$$

where $\boldsymbol{u}_j$ represents the solution of Eq. (1) under the virtual load $\boldsymbol{f}_j$ (i.e. $\boldsymbol{K}\cdot\boldsymbol{u}_j = \boldsymbol{f}_j$). The displacement change Δu_j can be simply calculated at one element level as [6–10,27–30]

$$\Delta u_j = \boldsymbol{u}_{ij}^{\mathrm{T}}\Delta\boldsymbol{K}_i\boldsymbol{u}_i = \boldsymbol{u}_{ij}^{\mathrm{T}}\cdot[\boldsymbol{K}_i(t) - \boldsymbol{K}_i(t-\Delta t)]\cdot\boldsymbol{u}_i, \tag{5}$$

where $\Delta\boldsymbol{K}_i$ is the change of the ith element's stiffness matrix due to the variation from the old thickness t to the next lower thickness $(t-\Delta t)$, $\boldsymbol{u}_i$ and $\boldsymbol{u}_{ij}$ denote the displacement vectors of the ith element under the real load $\boldsymbol{p}$ and the virtual unit load $\boldsymbol{f}_j$, respectively. The value,

$$\alpha_{ij} = \boldsymbol{u}_{ij}^{\mathrm{T}} \Delta \boldsymbol{K}_i \boldsymbol{u}_i \tag{6}$$

is defined as the *displacement sensitivity number* of the ith element, which is used to estimate the displacement change of the jth degree of freedom due to the thickness change of element i.

Under typical stiffness based criteria, structures are required to be stiff enough to carry the given loads. In structural design, strain energy is commonly considered as an inverse measure for the overall stiffness of a structure. It is well known that maximizing the overall stiffness is equivalent to minimizing the strain energy. In general, the reduction of element's thickness leads to the increase in strain energy and decrease in the stiffness. To evaluate the effect of the element thickness reduction on the overall strain energy of the structure, a *stiffness sensitivity number* can be further defined [7,8], similar to Eq. (6), as

$$\begin{aligned}\alpha_{s,i} &= \tfrac{1}{2}\boldsymbol{p}^{\mathrm{T}}\Delta\boldsymbol{u} = \tfrac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{K}^{-1}\,\Delta\hat{\boldsymbol{K}}_i\boldsymbol{u} \\ &= \tfrac{1}{2}\boldsymbol{u}_i^{\mathrm{T}}[\boldsymbol{K}_i(t) - \boldsymbol{K}_i(t-\Delta t)]\boldsymbol{u}_i = \tfrac{1}{2}\boldsymbol{u}_i^{\mathrm{T}}\,\Delta\boldsymbol{K}_i\boldsymbol{u}_i.\end{aligned} \tag{7}$$

2.2. Stress sensitivity number

In stress minimization, stress is taken into account as a design objective. Therefore, it is necessary to evaluate the stress change of an element due to the thickness variation in other elements. Without losing any generality, 2D plane stress problems are considered for simplicity. The derivation can be easily extended to other cases.

In finite element analysis, the kth element stress vector $\boldsymbol{\sigma} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}_k^{\mathrm{T}}$ can be calculated on the basis of its nodal displacement vector $\boldsymbol{u}^k = \{u_1, u_2, \ldots, u_j, \ldots, u_n\}_k^{\mathrm{T}}$ by

$$\boldsymbol{\sigma} = \boldsymbol{DBu}^k \tag{8}$$

where n is the total number of degrees of freedom for element k, $\boldsymbol{D}$ and $\boldsymbol{B}$ stress denote the conventional elastic and strain matrices respectively [27–29]. The change of the kth element stress vector due to the variation in the ith element's thickness can be expressed as

$$\Delta\boldsymbol{\sigma} = \{\Delta\sigma_{xx}, \Delta\sigma_{yy}, \Delta\sigma_{xy}\}_k^{\mathrm{T}} = \boldsymbol{DB}\,\Delta\boldsymbol{u}^k, \tag{9}$$

where vector $\Delta\boldsymbol{u}^k = \{\Delta u_1, \Delta u_2, \ldots, \Delta u_j, \ldots, \Delta u_n\}_k^{\mathrm{T}}$ denotes the nodal displacement change of the kth element. From the previous discussion, the change $\Delta u_j (j = 1, 2, \ldots, n)$ of the displacement component u_j can be calculated by the *displacement sensitivity number* as defined in Eq. (5) or Eq. (6), thereby Eq. (9) can be expressed as

$$\Delta\boldsymbol{\sigma} = \boldsymbol{DB}\{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ij}, \ldots, \alpha_{in}\}_k^{\mathrm{T}} = \boldsymbol{DB\alpha}, \tag{10}$$

where $\boldsymbol{\alpha} = \{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ij}, \ldots, \alpha_{in}\}_k^{\mathrm{T}}$ is hereafter referred to as the displacement sensitivity vector of the ith element, which indicates the displacement changes of all nodal components of the kth element due to the thickness variation of the ith element. Using Eq. (10), the change of the kth element stress can be calculated on the basis of the variations of its displacement components.

In engineering design, the stress level at each element may be measured by some sort of aggregate of all the stress components. For isotropic material, the von Mises stress is frequently used in this capacity. In a plane stress problem, the von Mises stress σ_{vm} at the kth element is expressed as [23]

$$\sigma_{\mathrm{vm}} = g(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = [\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{T\sigma}]_k^{1/2}, \tag{11}$$

where

$$\boldsymbol{T} = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is the coefficient matrix of von Mises quadratic form. The change in element's von Mises stress can be found by

$$\begin{aligned}\Delta\sigma_{\mathrm{vm}} &= \frac{\partial g}{\partial \sigma_{xx}}\cdot\Delta\sigma_{xx} + \frac{\partial g}{\partial \sigma_{yy}}\cdot\Delta\sigma_{yy} + \frac{\partial g}{\partial \sigma_{xy}}\cdot\Delta\sigma_{xy} \\ &= \nabla\boldsymbol{g}^{\mathrm{T}}\cdot\boldsymbol{DB\alpha} = \boldsymbol{\gamma}\cdot\boldsymbol{\alpha},\end{aligned} \tag{12}$$

where

$$\nabla\boldsymbol{g} = \left\{\frac{\partial g}{\partial \sigma_{xx}}, \frac{\partial g}{\partial \sigma_{yy}}, \frac{\partial g}{\partial \sigma_{xy}}\right\}_k^{\mathrm{T}}$$

is the gradient vector of von Mises stress function and $\boldsymbol{\gamma} = \nabla\boldsymbol{g}^{\mathrm{T}}\boldsymbol{DB} = \{\gamma_1, \gamma_2, \ldots, \gamma_j, \ldots, \gamma_n\}_k^{\mathrm{T}}$ represents the coefficients of the element displacement sensitivity vector $\boldsymbol{\alpha}$.

It has been previously shown, how to find the displacement change of the jth degree of freedom in element k by using a virtual unit load corresponding to this degree of freedom. Adopting this idea, to find the stress change of the kth element, there would need to be n virtual unit loads imposed to deal with each degree of freedom from Eqs. (10) and (12). In other words, the number of virtual systems depends on the total number of degrees of freedom per element. Thus, the computational cost will be considerably higher than that for a stiffness sensitivity analysis as Eq. (7). It would be beneficial to reduce the number of virtual systems and thereby improve the computational efficiency. According to the definition of the displacement sensitivity number in Eq. (6), Eq. (12) can be re-written as [9,30].

$$\Delta\sigma_{\mathrm{vm}} = \sum_{j=1}^{n}(\gamma_j\boldsymbol{u}_{ij}^{\mathrm{T}}\Delta\boldsymbol{K}_i\boldsymbol{u}_i) = \left(\sum_{j=1}^{n}\gamma_j\boldsymbol{u}_{ij}^{\mathrm{T}}\right)\cdot\Delta\boldsymbol{K}_i\cdot\boldsymbol{u}_i. \tag{13}$$

Note that $\boldsymbol{u}_{ij}^{\mathrm{T}}(j = 1, 2, \ldots, n)$ is the solution of the jth virtual system of

$$\boldsymbol{K}\boldsymbol{u}_{ij} = \boldsymbol{f}_j. \tag{14}$$

Multiplying Eq. (14) by the coefficient γ_j yields

$$\boldsymbol{K}\gamma_j\boldsymbol{u}_{ij} = \gamma_j\boldsymbol{f}_j. \tag{15}$$

Adding all the n virtual equilibrium equations given in Eq. (15), we have

$$\boldsymbol{K}\left(\sum_{j=1}^{n}\gamma_j\boldsymbol{u}_{ij}\right) = \sum_{j=1}^{n}\gamma_j\boldsymbol{f}_j \quad \text{or} \quad \boldsymbol{K}\tilde{\boldsymbol{u}}_{ik} = \tilde{\boldsymbol{f}}, \tag{16}$$

where $\tilde{\boldsymbol{u}}_{ik} = \sum_{j=1}^{n}\gamma_j\boldsymbol{u}_{ij}$ represents one virtual displacement vector and $\tilde{\boldsymbol{f}} = \sum_{j=1}^{n}\gamma_j\boldsymbol{f}_j$ denotes one virtual load. In a certain extent, the approach is similar to the traditional adjoint sensitivity analysis as in Refs. [27–29].

Eq. (16) introduces a new virtual system, where n non-zero components of the virtual load vector $\tilde{\boldsymbol{f}}$ are the corresponding coefficients γ_j instead of the unit value used in the previous displacement sensitivity analysis. The solution $\tilde{\boldsymbol{u}}_{ik}$ of this new virtual system (Eq. (16)) provides the result, which is required in Eq. (13). Consequently, Eq. (13) can be re-written as

$$\Delta\sigma_{\mathrm{vm}} = \tilde{\boldsymbol{u}}_{ik}^{\mathrm{T}} \cdot \Delta\boldsymbol{K}_i \cdot \boldsymbol{u}_i. \tag{17}$$

Thus, by introducing only *one* virtual load $\tilde{\boldsymbol{f}}$, the change of the von Mises stress at the kth element can be evaluated. The above processing significantly improves the computing efficiency. Therefore, we define

$$\alpha_{\sigma,i} = \tilde{\boldsymbol{u}}_{ik}^{\mathrm{T}} \cdot \Delta\boldsymbol{K}_i \cdot \boldsymbol{u}_i \tag{18}$$

as the ith element's *stress sensitivity number*, whose formulation is quite similar to those of the *displacement sensitivity number* in Eq. (6) and the *stiffness sensitivity number* in Eq. (7).

It is worth noting that the stress sensitivity number $\alpha_{\sigma,i}$ can be positive or negative, which implies that the stress of element k may increase or decrease, when there is a change in the thickness of element i. This provides a means of controlling the stress at a specified element by appropriate thickness design in other regions of a structure. The principle can be extended to any kind of structural modification.

It should also be pointed out that the above derivation is not restricted to the von Mises stress. Indeed, any form of stress function $\tilde{\sigma} = f(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ (for instance, principle stresses or Tresca stress) can have a corresponding stress sensitivity number, which can then be employed to evaluate the change of the relevant stress function due to thickness variations of other elements. Hence, different design requirements can be met.

As stated previously, the maximum stress in a structure plays a crucial role in an engineering design. It is often the major objective of a designer to minimize the maximum stress in a structure. One of the most important advantages of establishing the stress sensitivity number is that it identifies the location at which the structure should be modified, so that the maximum stress can be most efficiently reduced.

3. Evolutionary optimization procedure and implementation

There are two approaches to the design of thickness distribution of a structure. One is to set the initial design thickness at the maximum possible value, then gradually remove the redundant/inefficient material from the structure by progressive thickness reduction in terms of the stress sensitivity number. Another is to set the initial design thickness at a medium value, then gradually shift the material from the *over-designed* location to the *under-designed* location. These two different approaches are described separately in the following sections.

3.1. Morphing evolutionary structural optimization via material removal

The previous derivations have shown the possibility of minimizing the highest stress in a structure by appropriate thickness design. In a morphing ESO procedure, the initial design thickness is first set at the maximum possible value. Then by using the stress sensitivity number, redundant/inefficient material is removed from the structure by thickness reduction. This is performed in a discrete and progressive manner, i.e. from the old thickness t downwards to the next lower thickness $t - \Delta t$ at each iterative step. The change in stiffness matrix is

$$\Delta\boldsymbol{K}_i = \boldsymbol{K}_i(t) - \boldsymbol{K}_i(t - \Delta t), \tag{19}$$

where $\boldsymbol{K}_i(t)$ and $\boldsymbol{K}_i(t - \Delta t)$ denote the elemental stiffness matrices of the ith element for the old thickness t and the new thickness $(t - \Delta t)$, respectively. In practice, a set of allowable discrete values for element thicknesses is given in a decreasing order as

$$\boldsymbol{t}_{\mathrm{s}} = \{t_1, t_2, \ldots, t_r, t_{r+1}, \ldots, t_m\}^{\mathrm{T}}, \tag{20}$$

where $t_r > t_{r+1} (r = 1, 2, \ldots, m - 1)$ is the available design thickness and m is the number of discrete thickness; $\bar{t} = t_1$ and $\underline{t} = t_m$ are referred to as the upper bound and lower bound of the design variables, respectively. When the thickness of an element has reached the lower bound $\underline{t} = t_m$, it will no longer change. In accordance with different design requirements, the step sizes of Δt (i.e. $t_r - t_{r+1}$) may be constant or variable.

In the evolutionary procedure, the number of elements subjected to the thickness reduction should be appropriately small to ensure a smooth change between two iterations [6–10]. This is because the displacement sensitivity number is formulated by ignoring a high order term, and is valid only when the changes in the stiffness matrix and the displacement vector are small. In the ESO method, the number of elements with thickness reduction can be determined by the step size Δt and a prescribed *material removal ratio* (MRR), which is defined as the amount of removed material (volume) at each iteration over the total material (volume) of the initial design domain. A typical value for MRR is around 0.1–1%. This is the range that has been used to solve the example problems below.

The evolutionary iteration procedure for thickness reduction is given as follows:

Step 1: discretize the structure using a dense finite element mesh;

Step 2: define a variable thickness set $\boldsymbol{t}_s$ and the ESO driving parameters MRR;

Step 3: carry out a FEA for the real system as defined in Eq. (1);

Step 4: find element k with the maximum stress, then apply a virtual load $\tilde{\boldsymbol{f}}$ as given in Eq. (16) onto this element;

Step 5: perform another FEA for the virtual system as defined in Eq. (16);

Step 6: calculate stress sensitivity number $\alpha_{\sigma,i}$ using Eq. (18);

Step 7: reduce the thicknesses of a number of elements with the most negative $\alpha_{\sigma,i}$;

Step 8: repeat Steps 3–7 until a minimized maximum stress is found.

3.2. Morphing evolutionary structural optimization via material shifting

In the previous section, the optimization process is carried out by progressively removing material from the structure. Therefore, the total weight or the volume of the structure is gradually reduced with the evolutionary process. Sometimes, a designer may wish to improve the performance or reliability of a structure while keeping its weight (or volume) constant. This can be achieved by shifting material from the 'strongest' to the 'weakest' location. The concept of strength or weakness originated from the conventional fully stressed criterion [6,21,31], in which the material is gradually shifted from the least efficient (under-utilized) location to the most efficient (over-utilized) location. In stress minimization problems, the concept of 'strength' and 'weakness' needs to be distinguished between the case of thickness reduction from that of thickness increase [6]. The effect of the thickness variation of the ith element on the stress of the kth element is estimated by Eq. (17). In the case of thickness increase, we have

$$\Delta \boldsymbol{K}_i = \Delta \boldsymbol{K}_i^+ = \boldsymbol{K}_i(t) - \boldsymbol{K}_i(t + \Delta t) \tag{21}$$

and in the case of thickness reduction, we have

$$\Delta \boldsymbol{K}_i = \Delta \boldsymbol{K}_i^- = \boldsymbol{K}_i(t) - \boldsymbol{K}_i(t - \Delta t). \tag{22}$$

Therefore, two sensitivity numbers need to be calculated for each element, one for thickness increase

$$\alpha_{\sigma,i}^+ = \tilde{\boldsymbol{u}}_{ik}^{\mathrm{T}} \cdot \Delta \boldsymbol{K}_i^+ \cdot \boldsymbol{u}_i \tag{23}$$

and another for thickness reduction

$$\alpha_{\sigma,i}^- = \tilde{\boldsymbol{u}}_{ik}^{\mathrm{T}} \cdot \Delta \boldsymbol{K}_i^- \cdot \boldsymbol{u}_i. \tag{24}$$

To minimize the maximum stress, the most effective way is to remove the material from the elements with the most negative $\alpha_{\sigma,i}^-$ and then add it to the elements with the most negative $\alpha_{\sigma,i}^+$. When the evolving process has progressed to a certain degree, the lowest $\alpha_{\sigma,i}^-$ may be come positive. Then, the increase or reduction of the stress will depend on the deviation between the lowest $\alpha_{\sigma,i}^+$ and the lowest $\alpha_{\sigma,i}^-$. Usually, a tolerance τ needs to be prescribed for the convergence check on the target stress ($\sigma_{\max}$). If the relative change in the target stress in two successive iterations is less than the given tolerance τ, i.e.

$$\left| \frac{\sigma_{\max}^{\text{new}} - \sigma_{\max}^{\text{old}}}{\sigma_{\max}^{\text{new}}} \right| \leqslant \tau, \tag{25}$$

then, a convergent state for the target stress is reached and the evolutionary procedure may be terminated. To continue the iterations beyond such a convergent state will yield little or no improvement in the target stress.

In general, the variations in element stiffness matrix $\Delta \boldsymbol{K}_i^+$ and $\Delta \boldsymbol{K}_i^-$ due to the thickness change may be different. Therefore, they need to be calculated at the end of each finite element analysis. For the simplest plane stress problems, however, $|\Delta \boldsymbol{K}_i^+|$ is equal to $|\Delta \boldsymbol{K}_i^-|$. Under this condition, the calculation of the stress sensitivity number may be carried out just once.

The number of elements subjected to material shift is also determined by the step size Δt and the prescribed *material shift ratio* (MSR) [6,8], defined as the amount of material to be shifted at each iteration over the total material (volume) of the designed structure. Typical values for MSR are 0.1–1%, which have been used to solve the following examples.

In the evolution process, the thickness of original design is often set at a value between the upper bound $\bar{t}$ and lower bound $\underline{t}$ of the thickness set $\boldsymbol{t}_s$. When the thickness reaches its upper bound (i.e. $t = \bar{t}$) or lower bound (i.e. $t = \underline{t}$), respectively, they cannot change upwards or downwards any further, but may reverse direction if subsequent calculation of sensitivity numbers requires this. The evolutionary iteration procedure for minimizing the maximum stress while keeping the weight constant is organized as follows:

Step 1: discretize the structure using a dense finite element mesh;

Step 2: define a thickness set $\boldsymbol{t}_s$ and the ESO driving parameter MSR;
Step 3: perform a FEA for the real system as defined in Eq. (1);
Step 4: find the element k with the maximum stress, then add virtual load $\tilde{\boldsymbol{f}}$ as given in Eq. (16) onto this element;
Step 5: perform another FEA for the virtual system as defined in Eq. (16);
Step 6: calculate stress sensitivity numbers $\alpha_{\sigma,i}^{+}$ and $\alpha_{\sigma,i}^{-}$ using Eqs. (23) and (24);
Step 7: reduce the thickness of a number of elements with the lowest $\alpha_{\sigma,i}^{-}$ and increase the thickness of a number of elements with the lowest $\alpha_{\sigma,i}^{+}$ while keeping the volume constant;
Step 8: repeat Steps 3–7 until the objective has reached to a convergent state as defined by Eq. (25).

4. Examples and discussion

Based on the proposed stress minimization criterion, three examples are presented to demonstrate the capability of the ESO method. The first two examples have their volume progressively decreased by removing material, the third one maintains constant volumes by shifting material. All the examples are assumed to be under plane stress conditions and aimed at minimizing the maximum von Mises stress. To compare the proposed stress based designs with the more traditional stiffness based ones, each example is also solved in terms of the stiffness sensitivity numbers as defined in Eq. (7).

4.1. Example 1

A rectangular plate with the dimensions of 40 mm × 20 mm is clamped along two sides of the top edge and loaded by a uniform stress $\sigma = 10$ MPa along the middle region of the bottom edge as shown in Fig. 1. The regions of clamped and loaded elements are considered as the non-design domains. The Young modulus E and the Poisson ratio v are set at 210 GPa and 0.3, respectively. Due to the symmetry, only half of the region is analyzed with 20×20 four node plane stress elements. Upon completion of the evolutionary process, the symmetric half region is mirrored up to clarify the whole structure.

In this example, the design vector is set in equal intervals of $\Delta t = 0.1$ from 1.0 to 0.5 mm, i.e. $\boldsymbol{t}_s = \{1.0, 0.9, 0.8, 0.7, 0.6, 0.5\}^{\mathrm{T}}$. Initially, all elements are assigned the upper bound of the variable vector, i.e. $\bar{t} = 1.0$ mm. In the evolution process, a material removal ratio of $\mathrm{MMR} = 16 \times 0.1/190 = 0.84\%$ (i.e. 16 elements for each iteration) is adopted.

To minimize the maximum von Mises stress and to maximize the overall stiffness of the structure, the stress and stiffness based criterion are used in terms of the corresponding sensitivity numbers as defined in Eqs. (18) and (7), respectively. The evolution histories of the maximum von Mises stresses for both criteria are plotted in Fig. 2. For the case of stress minimization criterion, a notable reduction of the peak stress can be observed in the first few iterations. This shows that, at the early stage, the material removal is taking place in those elements with the most negative stress sensitivity number. After a few iterations, the thicknesses of all such elements have come to the design lower bound $\underline{t} = 0.5$. At this point, the objective stress reaches its minimum. Hereafter, the material removal is carried out on the positive sensitivity elements and consequently the peak stress gets increased progressively. This process can be clearly observed from the corresponding curve of the peak stress history in Fig. 2. The minimum value of the objective stress is identified at a volume ratio of 68%.

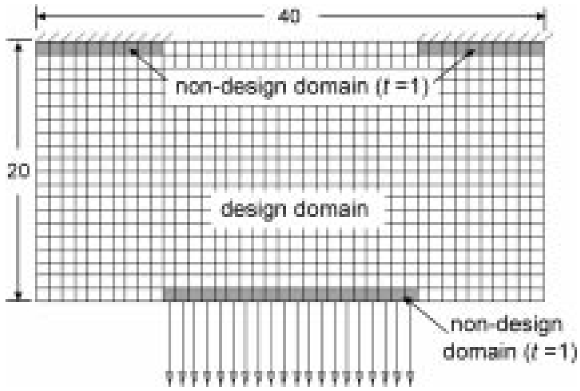

Fig. 1. Finite element model of initial design.

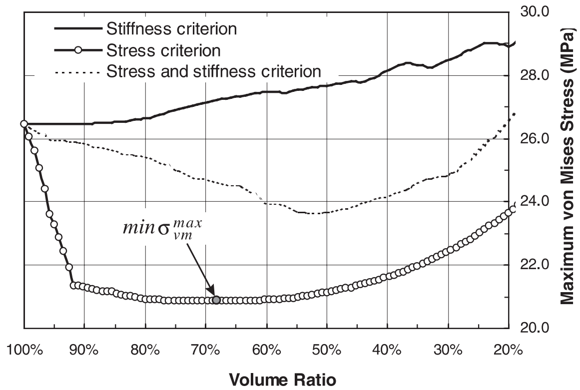

Fig. 2. Evolutionary histories of the maximum von Mises stresses.

When using the stiffness criterion, the optimal objective is to seek a stiffest design without any control on the maximum stress. For this reason, an increase in the peak stress cannot be avoided. From the corresponding stress evolution curve in Fig. 2, it can be seen that the maximum stress increases in some iteration steps and decreases at others. This implies that the elements with the least strain energy might not be those with negative stress sensitivity, or vice versa. In this sense, the stiffness criterion may not comply with the strength criterion, in other words, the 'stiffest design' may not be the 'strongest design'.

Fig. 3 shows the difference in the stiffness evolution histories for both criteria. As mentioned before, for a fixed load, the strain energy is the inverse measure of the overall stiffness of a structure. A structure is stiffer when its strain energy is lower. From the plotted evolutionary histories of the strain energy, one can see the strain energy of the 'stiffest design' (stiffness maximization) is considerably lower than that of the 'strongest design' (stress minimization). In the first few iterations, the strain energy is almost unchanged for the stiffness design but clearly increases for the strength design. This reveals the converse statement to the one above: the 'strongest design' may not be the 'stiffest design'.

In view of the difference between the stress criterion and the stiffness criterion, a weighting factor scheme is employed to produce a design, which is a compromise between these two criteria. In the ESO procedure, the sequence of material removal of an element is determined by that of its sensitivity number. In other words, the sequences of element sensitivity numbers are regarded as their relative contribution levels to design objectives. As a result, the element's overall contribution to both stress and stiffness criteria can be measured by adding the sequences of these two sensitivity numbers with weight factors as

$$S_i = w_\sigma \cdot S(\alpha_{i,\sigma}) + w_s \cdot S(\alpha_{i,s}), \tag{26}$$

where w_σ and w_s represent the weight factors for stress criterion and stiffness criterion, respectively, and usually: $w_\sigma + w_s = 1$. The weight factors can be used to emphasize the satisfaction of the different design criteria. In this example, considering that stress minimization is a major concern, it is assigned a larger weight factor as $w_\sigma = 0.7, w_s = 0.3$.

Fig. 4(a) and (b) shows the optimal designs for the stress minimization and the stiffness maximization, respectively. From a strength point of view, it is worth noting that the maximum von Mises stress is reduced by

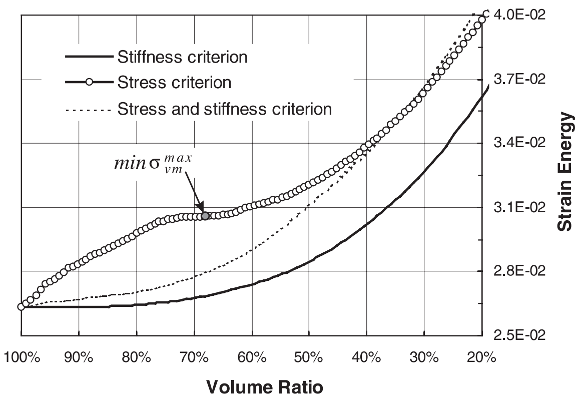

Fig. 3. Evolutionary histories of the strain energy.

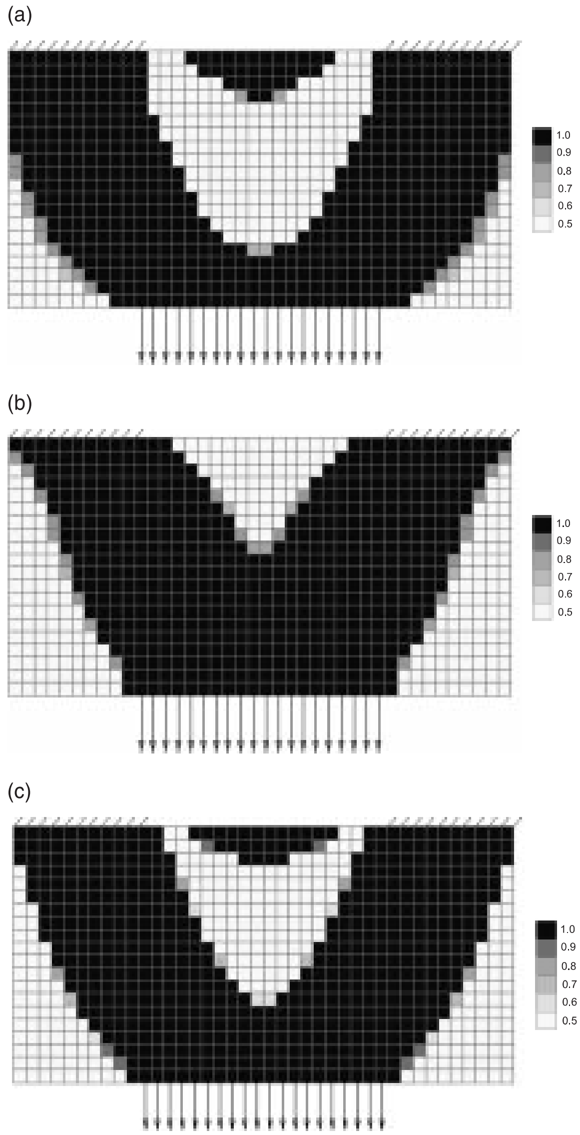

Fig. 4. Optimized designs based on different criteria ($V/V_0 = 68\%$) (a) Design for the stress minimization (material removed by 32%, σ_{vm}^{max} reduced by 21% and strain energy increased by 17%). (b) Design for the stiffness maximization (material removed by 32%, σ_{vm}^{max} increased by 3% and strain energy increased by 2%). (c) Design for stress minimization and stiffness maximization ($w_\sigma = 0.7, w_s = 0.3, \sigma_{vm}^{max}$ reduced by 7% and strain energy increased by 6%).

21% and the structure volume reduced by 32% for the stress criterion design. Compared to the stiffness based design with the same volume ratio of 68%, the stress reduction is remarkable and meaningful. This shows the most significant advantages of the proposed criterion over the conventional stiffness design, in which material volume and the maximum stress are simultaneously reduced. From the stiffness standpoint, however, the increase in the strain energy based on stress criterion reaches 17%, while that based on stiffness criterion is just 2%. Thus, there exists a notable stiffness difference between these two designs. Fig. 4(c) shows a trade off design between the stress minimization and the stiffness maximization, where the peak stress reduces by 7% and the strain energy increases by only 6% at the same volume ratio of 68%.

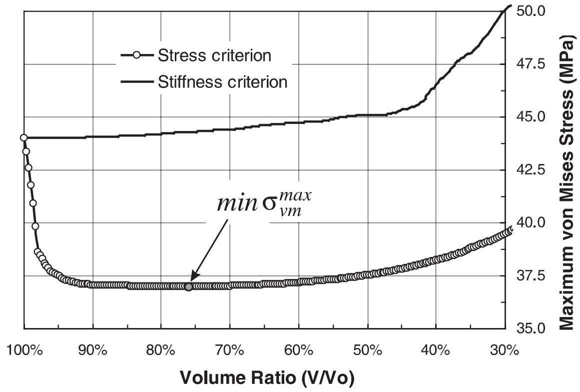

Fig. 6. Evolutionary histories of the maximum von Mises stresses.

4.2. Example 2

A region of 1000 mm × 300 mm is fully clamped at its two short edges and loaded by a uniform stress of 10 MPa on the top edge as illustrated in Fig. 5. To maintain the loading condition constant, the elements along the loading edge are considered as a non-design domain. The Young modulus E and the Poisson ratio ν are taken as 210 GPa and 0.3, respectively. Due to the symmetry, only half of the region is analyzed using a mesh of 40 × 24 four node plane stress elements. Upon completion of the evolutionary process, the symmetric half region is mirrored up to clarify the whole structure.

In this example, the discrete variable thickness set is given in an equal interval of $\Delta t = 1.0$ from 10 to 4 mm, i.e. $\boldsymbol{t}_s = \{10, 9.0, 8.0, 7.0, 6.0, 5.0, 4.0\}^T$, where there are totally seven discrete design thicknesses available. Initially, all elements in the design domain are assigned to the upper bound thickness $\bar{t} = 10$ mm and the elements in non-design domain are fixed at the lower bound thickness of $\underline{t} = 4$ mm. MRR $= 18/5520 \approx 0.2\%$ (18 elements) is adopted in the evolutionary processes.

Similar to the previous example, the evolutionary optimization is carried out, based on both the stress criterion and the stiffness criterion. To show the relation between the objective functions and material removal, the evolution histories are plotted with respect to the volume ratio of the structure. Shown in Fig. 6 is the evolution histories of the maximum von Mises stresses for the both criteria. Based on the stress criterion, the maximum von Mises stress in the structure is noticeably reduced as the material is removed. A minimum value of the objective stress can be found at a volume ratio $V/V_0 = 77\%$. However, such a stress reduction phenomenon cannot be observed in the stiffness based optimization, where the material removing process is driven by the stiffness sensitivity numbers, rather than the stress ones and thus the maximum stress is not controlled. The evolution histories of the strain energy are plotted in Fig. 7 for both criteria. It can be seen again that the increase in strain energy under the stress criterion is much faster than that under the stiffness criterion, in particular at the early stage of the material removal. These two illustrations of the evolutionary histories provide further evidence that stress minimiza-

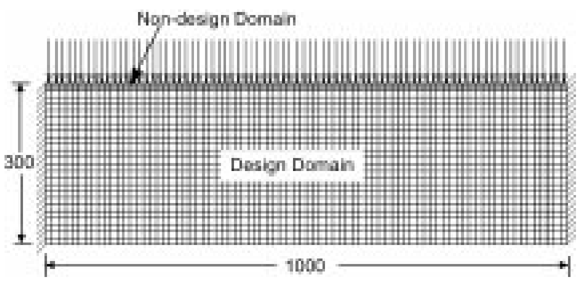

Fig. 5. Finite element model of initial design.

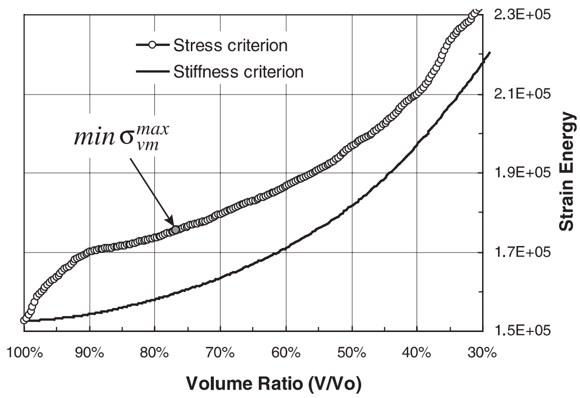

Fig. 7. Evolutionary histories of the strain energy.

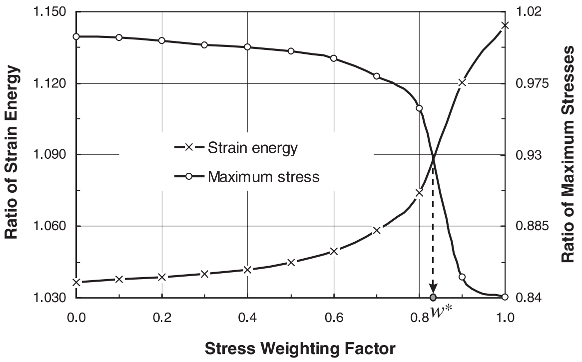

Fig. 8. Effects of weighting factor on the optimal objectives.

tion and stiffness maximization cannot be satisfied simultaneously.

To partially achieve both objectives, the weight factor scheme is again used in this example. In general, the effects of weighting factor w_σ and w_s on stress and stiffness objectives are not linear. This implies that a 50%–50% allocation of weights may not reflect an equal satisfaction of both criteria. Shown in Fig. 8 is the relation between stress weighting factor and the optimality criteria. It is clear that, with the increase in stress emphasis, the stress reduction becomes more significant, in other words, the structure gets stronger. But this leads to higher strain energy and the structure becomes more compliant. It is found that, at some stage, these two curves meet, which means an equal satisfaction of both objectives is achieved. In this example, a critical weighting factor of $w_\sigma = w^* = 0.83$ (or $w_s = 1 - w^* = 0.17$) can be identified for such requirement from Fig. 8. If the stress minimization is a major objective, the stress weighting factor should be assigned as $0.83 \leqslant w_\sigma \leqslant 1.0$, and if the stiffness maximization is a major objective, the stress weighting factor should be selected from $0 \leqslant w_\sigma \leqslant 0.83$.

Fig. 9(a) shows the optimized thickness distribution for the stress criterion, which corresponds to the point of minimum stress in Fig. 6. Comparing the optimized design with the initial model, it is found that the maximum von Mises stress is reduced by 17%, and at the same time, the weight/volume has also reduced by 23%.

In order to compare the thickness distributions under different criteria, Fig. 9(b) gives the stiffness based design, whose volume ratio is taken as the same as that in Fig. 9(a). The difference of the thickness designs can be clearly seen. In the stiffness design, the strain energy increases by 4% only, which is much less than that in the strength design (15%). Fig. 9(c) shows a trade off design between both criteria, where the peak stress reduces by 8% while the strain energy increases by 8% only. Obviously, such a design can compromise both optimal objectives and thus may be more practical.

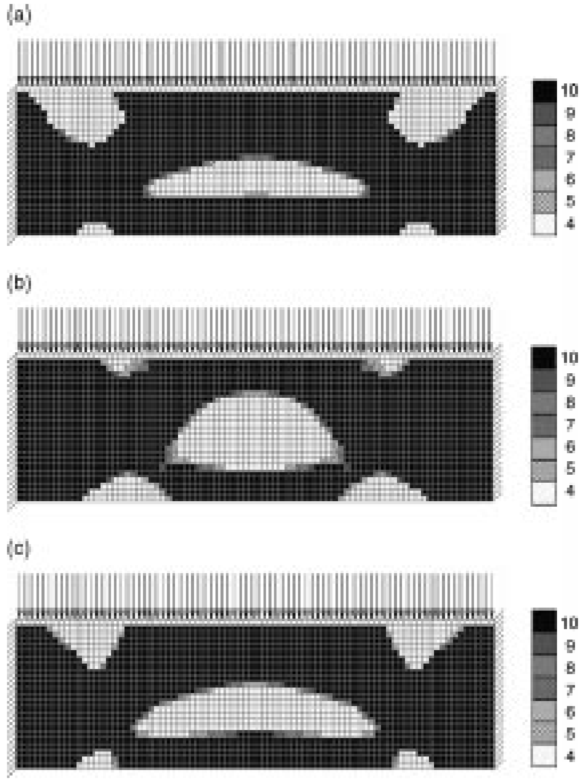

Fig. 9. Optimized designs based on the different criteria ($V/V_0 = 77\%$): (a) Design for the stress minimization (material removed by 23%, σ_{vm}^{max} reduced by 17% and strain energy increased by 15%). (b) Design for the stiffness maximization (material removed by 23%, σ_{vm}^{max} increased by 1% and strain energy increased by 4%). (c) Design for stress minimization and stiffness maximization ($w_\sigma = 0.83, w_s = 0.17, \sigma_{vm}^{max}$ reduced by 5%, strain energy increased by 8%).

4.3. Example 3: constant volume optimization through material redistribution

The previous two examples have the structural volume progressively reduced. For other situations, a designer may wish to re-distribute the material while keeping the volume constant. This example demonstrates the capability of the proposed method through material shifting techniques [6,8].

A 400 mm × 80 mm rectangular region clamped along two short edges is to be designed to support a point load of 1000 N, as illustrated in Fig. 10, as Ref. [11]. The Young modulus of $E = 210$ GPa and Poisson's ratio of $v = 0.3$ are taken for the material properties. For the finite element analysis, due to the symmetry,

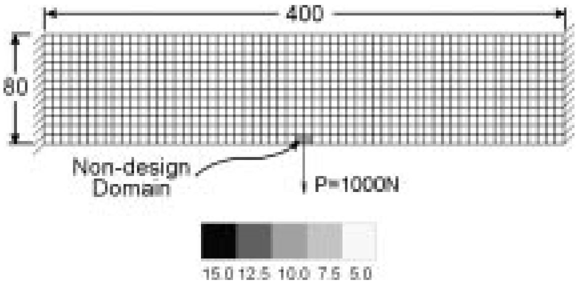

Fig. 10. Finite element model of initial design and thickness legend.

only half of the structure is analyzed using a mesh of 30×12 four node plane stress elements.

In this example, the discrete design variable is set in an equal interval of $\Delta t = 2.5$ from 15.0 to 5.0 mm, i.e. $\boldsymbol{t}_s = \{15.0, 12.5, 10.0, 7.5, 5.0\}^T$. Initially, all elements in design domain are assigned at $t = 10.0$ mm. The loaded elements (with the upper bound thickness of 15.0 mm) are considered as the non-design domain as illustrated in Fig. 10. In the evolving process, a $\text{MSR} = 1/120$ (12 elements) is prescribed.

Similar to the previous two examples, material shifting is carried out based on both the stress criterion and the stiffness criterion. To find a compromise design between these two criteria, the weight factors are selected as $w_\sigma = 0.4$ and $w_s = 0.6$ in this example.

Fig. 11 shows the evolution histories of the maximum von Mises stress for three different criteria. Unlike the preceding two examples, no minimum stress point can be found from the curve based on the pure stress criterion. At some stage, the lowest elemental sensitivity number come positive. Material removal from such elements would lead to increase in the target stress. On the

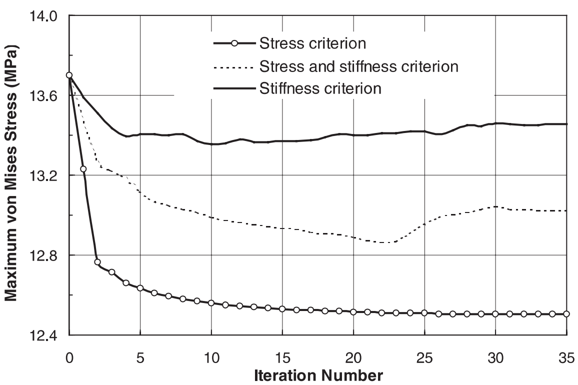

Fig. 11. Evolutionary histories of the maximum von Mises stresses.

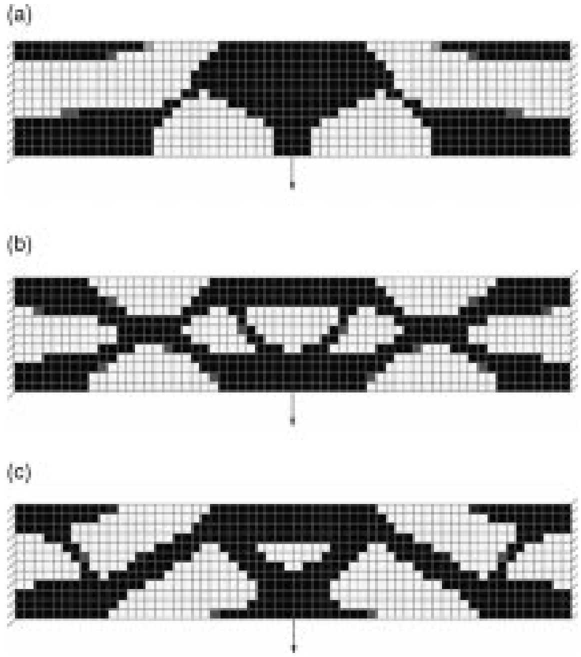

Fig. 12. Optimized designs based on different criteria. (a) Design for the stress minimization. (b) Design for stiffness maximization. (c) Design for stress minimization and stiffness maximization.

other hand, the material is added onto the highest sensitivity elements. This results in the decrease in the target stress. It is found that the sum of the elemental sensitivities with the material increase is always greater than that with the material decrease. For this reason, the stress criterion steadily makes the maximum stress reduced. However, the stiffness criterion cannot guarantee this. From Fig. 11, one also can see that the multiple criteria by the weights of $w_\sigma = 0.4$ and $w_s = 0.6$ can result in a balance between both sub-objectives.

The pictures shown in Fig. 12(a)–(c) are the optimized material distributions for the stress, stiffness and the combined criteria. It is found that the material distributions produced by the stress and stiffness criteria are similar to those obtained by Yang and Chen [11] using a 3D material density method.

5. Concluding remarks

In this study, a discrete stress sensitivity number is proposed. According to the sign of the stress sensitivity number, a structure can be divided into two regions. Removing material in the negative sensitivity region or adding material in positive sensitivity region will lead to a stress reduction on the specific elements. The calculation of the stress sensitivity number by introducing an adjoint virtual system does not increase the computational cost to any great extent.

Optimal thickness design, based on the proposed evolution criterion of stress sensitivity number, offers a new capability for structural design aimed at minimizing the maximum stress. The studies in this paper show the successful application of the present criterion in discrete thickness design. Through the numerical examples, it has been demonstrated that the reduction of both stress and material weight may be achieved simultaneously.

In this paper, the design for stress minimization is compared with that for the stiffness maximization. It is found that the optimized thickness distributions aimed at the stress minimization may differ from those aimed at stiffness maximization. The 'stiffest design' may not mean the 'strongest design', and *vice versa*. Appropriate trade-off between them can be made by a weighted average of the two sensitivities. The weighting factors can be assigned in accordance to different design objectives and emphases. In certain instances, it is significant to find a design that best compromises both criteria concurrently.

Compared with other mathematical programming based methods, the proposed ESO method offers significant simplicity and effectiveness. It follows a very simple iterative cycle of finite element analysis, calculation of sensitivity numbers and evolutionary thickness re-distribution, which avoids tedious sub-iterations of classical non-linear programming. The method itself does not involve any complicated mathematical operations, nor does it need to know more details than the elemental stiffness matrix. It is convenient and suitable to integrate the proposed method into any existing finite element program, where end users usually have no access to source codes. This provides engineers with a very simple and practical alternative design tool.

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